Irreducibility and π_2 (Lecture 27)

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In the last lecture, we introduced the notion of an irreducible 3-manifold: a 3-manifold M is said to be irreducible if every embedded 2-sphere in M bounds a disk (on exactly one side). Our stated motivation was that embedded 2-spheres were good candidates to represent nontrivial classes in $\pi_2 M$. Our first goal in this lecture is to show that this is indeed the case.

Proposition 1. *Let M be a 3-manifold, and let $S \hookrightarrow M$ be an embedded 2-sphere. The following conditions are equivalent:*

(1) *The sphere S bounds a disk in M.*

(2) *The sphere S represents a trivial class in $\pi_2 M$.*

Remark 2. The statement of Proposition 1 is a little sloppy: the homotopy group $\pi_2 M$ is really only well-defined after we have chosen a base point on M. If M is connected, then the groups $\pi_2(X, x)$ and $\pi_2(X, y)$ can be related by choosing a path from x to y, but the identification depends on this choice of path via the action of $\pi_1 M$ on $\pi_2 M$. This means that the class of S in $\pi_2 M$ is only well-defined up to the action of $\pi_1 M$; however, the condition that this class vanishes is invariant under the action of $\pi_1 M$ (the vanishing is equivalent to the requirement that $S \hookrightarrow M$ is homotopic to a constant map, ignoring the base points).

Proof. (In what follows, we do not assume that M is compact.) It is clear that if S bounds a disk, then S is nullhomotopic. Conversely, suppose that S is nullhomotopic. Suppose first that M is simply connected. Since $[S] = 0 \in \mathrm{H}_2(M; \mathbf{Z}/2\mathbf{Z})$, the 2-sphere S is separating (though the converse can fail in the noncompact setting); we can therefore write $M = M_0 \coprod_{S^2} M_1$ where M_0 and M_1 are 3-manifolds with 2-sphere boundary. We have an exact sequence

$$\mathrm{H}_2(S) \xrightarrow{j} \mathrm{H}_2(M_0) \oplus \mathrm{H}_2(M_1) \to \mathrm{H}_2(M)$$

(all homology computed with $\mathbf{Z}/2\mathbf{Z}$ coefficients). Since $[S]$ vanishes in $\mathrm{H}_2(M)$, we deduce that the class $([S], 0)$ lies in the image of j: in other words, either $([S], 0)$ or $(0, [S])$ vanishes. Assume the former, and let $\widehat{M_0}$ be the 3-manifold obtained from M_0 by capping off the boundary sphere. We have an exact sequence

$$\mathrm{H}_3(\widehat{M_0}) \to \mathrm{H}_2(S^2) \xrightarrow{i} \mathrm{H}_2(D^3) \oplus \mathrm{H}_2(M_0).$$

Since the map i is not injective, we deduce that $\mathrm{H}_3(\widehat{M_0})$ is nonzero. By Poincare duality (the simple connectivity of $\widehat{M_0}$ guarantees orientability), we deduce that $\mathrm{H}^0_c(\widehat{M_0})$ does not vanish, so that $\widehat{M_0}$ is a compact, simply connected 3-manifold. By the Poincare conjecture, $\widehat{M_0}$ is a 3-sphere, so that M_0 is a disk bounded by S.

Suppose now that M is not simply connected; we still have $M = M_0 \coprod_S M_1$ as above. Let $\widetilde{M}$ be a universal cover of M, and $\pi : \widetilde{M} \to M$ the projection map. Since S is simply connected, we can lift S to a 2-sphere $\widetilde{S}$ in $\widetilde{M}$. Since $\pi_2 M \simeq \pi_2 \widetilde{M}$, the sphere $\widetilde{S}$ is nullhomotopic and therefore bounds a disk. This disk might contain other preimages of S: however, by adjusting our choice of $\widetilde{S}$ we can arrange that $\widetilde{S}$ contains a disk D which intersects the inverse image of $\pi^{-1} S$ only in $\widetilde{S}$. It follows that $\pi(D) \subseteq M_0$ or $\pi_D \subseteq M_1$;

without loss of generality we may assume the former. The map π induces a local homeomorphism $D \to M_0$. Since D is compact, this local homeomorphism is proper, and is therefore a finite-sheeted covering space. Since the Euler characteristic of D is 1, this covering space has 1-sheet so that $M_0 \simeq D$ is a disk bounded by the sphere S, as required. □

It follows that if a compact 3-manifold M is not irreducible, then $\pi_2 M$ does not vanish. We might ask if the converse is true: if $\pi_2 M$ is nonvanishing, does M fail to be irreducible? The answer is not obvious: the nonvanishing of $\pi_2 M$ guarantees a nontrivial homotopy class of map $i : S^2 \to M$, but the map i need not be an embedding. However, it turns out that the existence of nontrivial homotopy class guarantees the existence of an embedded 2-sphere with a nontrivial homotopy class, at least when M is oriented.

Theorem 3 (The Sphere Theorem)**.** *Let M be an oriented 3-manifold, and suppose that $\pi_2 M$ is nontrivial. Then there exists an embedded 2-sphere $S \hookrightarrow M$ representing a nontrivial class in $\pi_2 M$. More generally, given any $\pi_1 M$-invariant normal subgroup $N \subset \pi_2 M$, there exists an embedded 2-sphere $S \hookrightarrow M$ representing an element of $\pi_2 M$ which does not belong to N.*

We will prove this theorem over the course of the next few lectures. The idea is to begin with an arbitrary map $i : S \to M$ representing a homotopy class which does not belong to N, and to adjust this map to make it an embedding. The same techniques will be used to prove the following companion to the sphere theorem:

Theorem 4 (The Loop Theorem)**.** *Let M be a 3-manifold with boundary and let X be a boundary component of M. If N is a normal subgroup of $\pi_1 X$ which does not contain the kernel of the map $\pi_1 X \to \pi_1 M$, then there exists an embedding $(D^2, S^1) \to (M, X)$ such that the loop $S^1 \hookrightarrow X$ represents a class in $\pi_1 X$ which does not belong to N.*

Remark 5. The hypothesis of orientability in the sphere theorem is essential. If P denotes the 2-dimensional real projective space, then $P \times S^1$ is a nonorientable 3-manifold with $\pi_2(P \times S^1) \simeq \mathbf{Z}$, yet $P \times S^1$ does not contain any nontrivial embedded 2-spheres (it contains many *immersed* 2-spheres, given by the double covering $S^2 \to P$).

We now begin to pave the way for our proofs of the loop and sphere theorems by discussing the notion of a *general position* map from a surface S into a 3-manifold M. We will treat this notion informally and not give a precise definition: roughly speaking, a map $i : S \to M$ is in general position if the behavior of i satisfies all of the conditions we like that can be guaranteed by moving the map i by a small amount. In particular, any "singularities" of the map i can be assumed to appear in the expected codimension, which means they do not appear at all if the expected codimension is ≥ 3 (in S) or ≥ 4 (in M):

Assume therefore that we are given a smooth map $i : S \to M$. How can this map fail to be an embedding? There are essentially two things that can go wrong:

(i) The map i can be fail to be an immersion at a point $s \in S$. In other words, the derivative Di can fail to have rank 2 at s. The derivative Di_s takes values in the 6-dimensional space of linear maps $T_{S,s} \to T_{M,i(s)}$. A linear map of rank 1 is determined by specifying a 1-dimensional quotient Q of $T_{S,s}$ (the set of such choices forms a 1-dimensional space), a 1-dimensional subspace Q' of $T_{M,i(s)}$ (where we have a 2-dimensional space of choices), and a linear isomorphism $Q \simeq Q'$ (for which we have 1-dimensional space of choices); in total, we find that the space of maps having rank 1 is a manifold of dimension $1+2+1 = 4$. Including the zero map does not increase the dimension: we conclude that Di_s should be expected to have rank ≤ 2 in on a subset of S having codimension 2. Since S is a surface, the map i should fail to be an immersion at a discrete set of points of S. The images of these points in M are called *branch points* of the map i.

(ii) The map i can fail to be injective, so that $i(x) = i(y)$ for $x \neq y$. Since $i(x)$ and $i(y)$ take values in the 3-manifold M, we should expect the relation $i(x) = i(y)$ to hold with codimension 3 among $(x, y) \in S^2$. We will say that $x \in M$ is a *double point* of i if $i^{-1}\{x\}$ has cardinality 2. If i is in general position, then we expect the set of double points to be a smooth submanifold of codimension 1 in M. We can also arrange that this set does not intersect the set of branch points (although, as we will see in a moment, every branch point lies in the *closure* of the set of double points).

(*iii*) The map i can fail to be injective more drastically: we can have $i(x_1) = i(x_2) = \ldots = i(x_n)$. This behavior is to be expected in codimension $3(n-1)$ in the space S^n of dimension $2n$. If $n > 3$, then $3(n-1) > 2n$ so that a generic map i will have not exhibit this behavior. If $n = 3$, then we expect this to happen for a discrete subset of S^3: in other words, we expect an isolated set of points $x \in M$ where $i^{-1}\{x\}$ has cardinality 3. We will call such points *triple points* of the map i.

What does the map i look like near a branch point? If we work in the piecewise linear category, then the local structure of a PL map $i : D^2 \to D^3$ is given by taking the cone over some PL map $i_0 : S^1 \to S^2$. If i_0 is an embedding, then so is i, and we do not have any branching. We may therefore assume that i_0 fails to be an embedding and therefore has some double points. It follows that every branch point of i lies at the endpoint of a curve of double points of i. (For a generic choice of i, the curve $i_0 : S^1 \to S^2$ will have only a single self-intersection so that this double curve is unique. However, we will not need to know this.)