Product Structure Theorem: End of the Proof (Lecture 22)

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We continue our proof of the product structure theorem for smooth structures on PL manifolds. Recall that we are reduced to proving the following:

Proposition 1. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \rightarrow \mathbb{R}$ denote the projection onto the last factor. Let $f : K \rightarrow \mathbb{R}^{m+1}$ be a PD embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$ and $f(0) = 0$.*

(2) *For $.8 \leq t \leq 1$ and $x \in \partial K$, we have $f(tx) = tf(x)$.*

(3) *The projection π is injective when restricted to the vertices of K (with respect to some PL triangulation), so that $\pi \circ f^{-1}$ is regular on the interior of the unit ball except possibly at the origin.*

(4) *The map $\pi \circ f$ coincides with π on $\pi^{-1}(-\epsilon, \epsilon) \cap S^m$ for ϵ sufficiently small.*

(5) *The map f is PL in a neighborhood of the origin.*

Then, after modifying f by a PD isotopy which is trivial on ∂K, we can arrange that $\pi \circ f^{-1}$ is regular on the interior of the unit ball.

Replacing f by its restriction to tK for t close to 1, we can assume that $\pi \circ f$ is regular on $B(1) - \{0\}$. Let $C_0 = \partial K \cap \pi^{-1}[-\epsilon, \epsilon]$ and let $C = [.8, 1] \times C_0 \subseteq K$. Conditions (4) and (2) guarantee that $\pi | C = (\pi \circ f) | C$. Let $D \subseteq K$ be a PL neighborhood of the origin on which f is PL. Choose a triangulation S of K with the following properties:

(1) The subpolyhedra C and D of K are unions of simplices.

(2) The map π is injective on the vertices of K.

Let Lf denote the linearized version of f with respect to the triangulation S (that is, the unique map which is linear on each simplex of S and which agrees with f on vertices). Choose a PL function $\chi : K \rightarrow [0, 1]$ such that $\chi = 1$ on $.8K$ and $\chi = 0$ on $[.9, 1] \times \partial K$, and define a homotopy $\{f_t : K \rightarrow \mathbb{R}^{m+1}\}$ by the formula

$$f_t(x) = t\chi(x)Lf(x) + (1 - t\chi(x))f(x).$$

We have seen that if S is a sufficiently fine triangulation, then f_t is a PD isotopy from f to f_1, where f_1 is a map which is PL on $.8K$ and agrees with f on $[.9, 1] \times \partial K$. Since f is already PL on D, we have $f = f_1$ on D, so that $\pi \circ f_1^{-1}$ is regular on $f_1(D - \{0\})$. Similar reasoning shows that $\pi \circ f_1 = \pi \circ f = \pi$ on $C \subseteq K$. Choosing S sufficiently fine, we can arrange that f_1 is an arbitrarily close approximation to f (in the C^1-sense). In particular, we can arrange that:

(a) The map $\pi \circ f_1^{-1}$ is regular on $B_1 - f_1(D)$ (and therefore on $B(1) - \{0\}$).

(b) For every point $x \in f_1(.8K)$, we have $tx \in f_1(.8K)$ for $0 \le t \le 1$ (since $f_1(.8K)$ closely approximates $f(.8K)$, which is the ball $B(.8)$).

(c) For $x \notin C$, we have $|(\pi \circ f)(x)| \ge \frac{\epsilon}{2}$.

We define another map $f_2 : K \to \mathbb{R}^{m+1}$ so that for $x \in \partial K$, we have

$$f_2(tx) = \begin{cases} f_1(tx) & \text{if } .8 \le t \le 1 \\ \frac{t}{.8} f(.8x) & \text{if } 0 \le t \le .8. \end{cases}$$

Using the assumption that $\pi \circ f_1^{-1}$ is regular on $B(1) - \{0\}$, it is easy to check that $\pi \circ f_2^{-1}$ is regular on $B(1) - \{0\}$ (if $v \in \mathbb{R}^{m+1}$ is a regular vector for $\pi \circ f_1^{-1}$ at a point $x \in f_1(.8K)$, then v is regular for $\pi \circ f_2^{-1}$ at tx for $t \in (0, 1]$). In order to proceed, we need to know the following:

Claim 2. *There exists a PD isotopy from f_1 to f_2, fixed near ∂K.*

In fact, there exists a PL isotopy from f_1 to f_2 which is supported on $.8K$. This is an obvious consequence of the following result:

Theorem 3 (The Alexander Trick)**.** *Let $\phi, \phi' : D^n \to D^n$ be two PL homeomorphisms from the PL n-disk to itself. If ϕ and ϕ' agree on the boundary ∂D^n, then ϕ is PL isotopic to the identity.*

Composing with an inverse to ϕ', we are reduced to proving that if ϕ is the identity on ∂D^n, then ϕ is PL isotopic to the identity. We will give a proof in the topological category: the PL version of Theorem 3 can be established using a construction of the same flavor. Let us identify D^n with the unit ball $B(1) \subseteq \mathbb{R}^n$. We define an isotopy $\{\phi_t : B(1) \to B(1)\}$ by the formula

$$\phi_t(sx) = \begin{cases} sx & \text{if } t \le s \\ t\phi(\frac{sx}{t}) & \text{if } t > s. \end{cases}$$

where $x \in \partial B(1)$. It is easy to see that ϕ_t is an isotopy from $\phi_0 = \text{id}$ to $\phi_1 = \phi$.

Remark 4. The Alexander trick does not work in the smooth category; the map described above exhibits essential nondifferentiable behavior when $t = 0$.

We now return to the proof of Proposition 1. Note that f_2 has the following properties:

- If $x \in C_0 \subseteq \partial K$, then $\pi f_2(x) = \pi(x)$.
- If $x \in \partial K - C_0$, then $|(\pi \circ f_2)(tx)| \ge \frac{t\epsilon}{2}$.

We are free to replace f by f_2. Since $\pi \circ f_2^{-1}$ is regular away from the origin, we are free to replace K by any smaller neighborhood of the identity. In particular, we can replace K by the star of the origin with respect to some triangulation of $.8K$ with respect to which $f_2|.8K$ is PL. We are thereby reduced to proving the following version of Proposition 1

Proposition 5. *Let $K \subseteq \mathbb{R}^m \times \mathbb{R}$ be a polyhedron which is the closed star of the origin 0 with respect to some PL triangulation of $\mathbb{R}^{m+1}$ (so that K is the cone on ∂K, with the origin as the cone point), let $\pi : K \to \mathbb{R}$ denote the projection onto the last factor. Let $f : K \to \mathbb{R}^{m+1}$ be a PL embedding satisfying the following conditions:*

(1) *The image of f is the unit ball $B(1) \subseteq \mathbb{R}^{m+1}$ and $f(0) = 0$.*

(2) *The projection π is injective when restricted to the vertices of K.*

(3) *There exists a subpolyhedron $C_0 \subseteq \partial K$ and a constant ϵ such that $|\pi(tx)|, |\pi \circ f(tx)| \ge t\epsilon$ for $x \notin C_0$.*

(4) *The maps* $\pi \circ f$ *and* π *agree on* C_0 *(and therefore on the cone* $\overline{C} = \{tx : x \in C_0, t \in [0,1]\}$*).*

Then, after modifying f *by a PD isotopy which is trivial on* ∂K*, we can arrange that* $\pi \circ f^{-1}$ *is regular on the interior of* $f(K)$*.*

We will construct a PD isotopy $\{f_t\}$ of f with the following properties:

(i) For every simplex σ of our triangulation of K, the $\{f_t|\sigma\}$ is a smooth isotopy from σ to $f(\sigma)$.

(ii) The isotopy $\{f_t\}$ is fixed on ∂K.

(iii) We have $\pi \circ f_1 = \pi$ in a neighborhood of the origin.

Since π is injective on the vertices of K, the map $\pi \circ f_1^{-1}$ will automatically be regular on the interior of K except possibly at the origin; condition (iii) will guarantee regularity at the origin as well. It therefore suffices to construct $\{f_t\}$. Since π is injective on the vertices of K, the set V of vertices of ∂K can be partitioned into two subsets $V_+ = \{v \in V : \pi(v) > 0\}$ and $V_- = \{v \in V : \pi(v) < 0\}$. Refining our triangulation of ∂K if necessary, we may assume that every simplex τ of ∂K which contains vertices from both V_+ and V_- belongs to C_0. For each simplex τ of ∂K, let $\widehat{\tau}$ denote the cone of this simplex (with cone point the origin). We construct the isotopies $\{f_t|\widehat{\tau}\}$ one simplex at a time. If τ is a simplex of C_0, then we let $\{f_t|\widehat{\tau}\}$ be the trivial isotopy (this satisfies (iii) since f satisfies (4)). Otherwise, we may assume without loss of generality that each vertex v of τ belongs to V_+. Let $v_1, \ldots, v_k$ be the vertices of τ. There exist positive constants $\{a_i\}_{1 \leq i \leq k}$ such that $\pi(v_i) = a_i(\pi \circ f)v_i$. We define a homotopy $\{g_t : \widehat{\sigma} \to \mathbb{R}_{\geq 0}\, f(\widehat{\sigma})\}$ by the formula

$$g_t(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \sum \lambda_k f(v_i)(ta_i + (1-t)).$$

Then g_t is a homotopy from $f|\widehat{\tau} = g_0$ to a map g_1 satisfying $\pi \circ g_1 = \pi$. Note that g_t carries a neighborhood of the origin in $\widehat{\tau}$ into $f(\widehat{\tau})$. Using a relative version of the smooth isotopy extension theorem, we can find an isotopy $\{f_t|\widehat{\tau} \to f(\widehat{\tau})\}$ which is supported in a compact subset of $\widehat{\tau} - \tau$, agrees with g_t near the origin, and agrees with the isotopies we have already constructed on the cone of $\partial \tau$.