Product Structure Theorem: Isolating Singularities (Lecture 20)

March 30, 2009

In this lecture, we will continue our efforts to prove the product structure theorem. As in the last lecture, we will be content to treat the special case where the set K is empty, and the product is with $\mathbb{R}$ rather than with $[0, 1]$. In the last lecture, we reduced this to proving the following assertion:

Proposition 1. *Let M be a PL manifold, and suppose we are given a compatible smooth structure on $X = M \times \mathbb{R}$. Let $\pi : X \to \mathbb{R}$ denote the projection onto the second factor (so that π is a PD map). Then, after altering the smooth structure on X by a PD isotopy, we can arrange that the map π is regular.*

To prove this, it is useful to have a criterion for testing whether or not a map is regular. Fix a smooth triangulation of X for which π is PL (and therefore smooth) on each simplex. Let $x \in X$, and let σ denote the simplex containing x in its interior. The tangent space $T_{X,x}$ to X at x contains the tangent space $T_{\sigma,x}$ as a linear subspace. Let $v \in T_{\sigma,x}$. Note that every simplex τ containing x contains σ, so the derivatives $D_v(\pi|\tau)$ all agree with $D_v(\pi|\sigma)$. It follows that $D_v(\pi) = D_v(\pi|\sigma)$. It follows that π is regular at x unless the derivative of $\pi|\sigma$ is identically zero. We have proven:

Lemma 2. *If $x \in X$ is a point where π is not regular and σ is as above, then σ lies in a fiber of π.*

Corollary 3. *Fix a triangulation of the polyhedron $X \simeq M \times \mathbb{R}$, and suppose that the restriction of π to the set of vertices of this triangulation is* injective. *Then π is regular away from the set of vertices of the triangulation. In particular, π is regular away from an isolated set of points.*

We can always arrange to be in the situation of Corollary 3. To see this, choose any triangulation of $M \times \mathbb{R}$ which is sufficiently fine that the star of each vertex has a neighborhood with a PL product chart $\mathbb{R}^m \times \mathbb{R}$. For each vertex v, let $L(v)$ denote the link of v and $\mathrm{St}(v)$ its star. We define a PL isotopy h_t of $M \times \mathbb{R}$, supported in the star $\mathrm{St}(v)$, which we view as a closed subset of $\mathbb{R}^m \times \mathbb{R} \simeq \mathbb{R}^{m+1}$. Fix $v' \in \mathbb{R}^{m+1}$. For each $t \in [0, 1]$, there is a unique map $h_t : \mathrm{St}(v) \to \mathbb{R}^m \times \mathbb{R} \subseteq M \times \mathbb{R}$ which is linear on each simplex, the identity on $L(v)$, and carries v to $(1-t)v + tv'$. If v' is chosen sufficiently close to v, then this defines a PL isotopy of M, where h_1 moves v to v'. We can assume that $\pi(v')$ is distinct from $\pi(w)$, for any other vertex w of the triangulation. Applying this construction repeatedly and concatenating the resulting isotopies (note that only finitely many isotopies have support near any fixed point of $M \times \mathbb{R}$, so the concatenation is well-defined), we can arrange that π is injective when restricted to vertices, as desired.

We may now assume that π is regular away from the set of vertices with respect to some smooth triangulation of X. We would like to adjust the smooth structure on X by a PD isotopy to arrange that π is everywhere regular. Since the set of vertices of X is isolated, it will suffice to construct these isotopies one vertex at a time. More precisely, we will prove the following:

Proposition 4. *Let v be a vertex with respect to some smooth triangulation of X, and let K denote the star of v. Assume that π is injective on vertices of X, so that π is regular on the interior of K except perhaps at v. Then it is possible to alter the smooth structure on X by means of a PD isotopy supported on the interior of K, so that π is regular on the whole interior of K.*

Applying this proposition repeatedly and concatenating the resulting isotopies, we will obtain a proof of Proposition 1. We are therefore reduced to proving Proposition 4. Moreover, we may assume without loss of generality that our triangulation of X is sufficiently fine that the star of each vertex is contained in a PL product chart $\mathbb{R}^m \times \mathbb{R}$ and also a smooth chart. We will identify K with its image in $\mathbb{R}^{m+1}$. Without loss of generality, we may assume that $v \mapsto 0$, so that K can be identified with the cone on the link $L(v) = \partial K$, which is an m-sphere equipped with a PL embedding into $\mathbb{R}^{m+1} - \{0\}$. The map $\pi | K : K \to \mathbb{R}$ is given by projection onto the $(m+1)$st coordinate. As above, we may assume that π is injective on vertices. In particular, $\pi(w) \neq 0$ whenever w is a vertex of $L(v)$.

The smooth structure on X is given by a PD embedding $f : K \to \mathbb{R}^{m+1}$. We wish to modify f by a PD isotopy which is the identity near ∂K, so that the map $\pi \circ f^{-1} : f(K) \to \mathbb{R}$ is regular on the interior of K.

We can therefore rephrase our problem as follows:

Problem 5. *Let $K \subseteq \mathbb{R}^{m+1}$ be a polyhedron which is the cone (with cone point 0) on its boundary ∂K, let $\pi : K \to \mathbb{R}$ be projection onto the last factor, and assume that π is injective on the vertices of K. Let $f : K \to \mathbb{R}^{m+1}$ be a PD embedding, and assume $f(0) = 0$. Then, after adjusting f by a PD isotopy which is fixed near ∂K, we can arrange that $\pi \circ f^{-1}$ is regular on the interior of $f(K)$.*

Remark 6. In the course of solving Problem 5, we are free to replace K by its image rK for $r \in (0,1)$: any PD isotopy of $f|rK$ can then be extended to a PD isotopy of f by declaring it to be the identity on $K - rK$.

Our first step is to "linearize" the map f. Since f is differentiable on each simplex of K, we can define a map $f' : K \to \mathbb{R}^{m+1}$ which is linear on each simplex by taking the derivatives of f at the origin. There is a PD homotopy from f' to f, given by the formula

$$f_t(x) = \begin{cases} t^{-1} f(tx) & \text{if} t \neq 0 \\ f'(x) & \text{if } t = 0. \end{cases}$$

This homotopy is generally not trivial on the boundary ∂K. To fix this, choose a smooth map $\chi : K \to [0,1]$ which is supported in a small neighborhood U of the origin, such that χ is identically equal to 1 in an open set $V \subseteq U$ containing 1, and define

$$g_t(x) = \chi(\frac{x}{N}) f_t(x) + (1 - \chi(\frac{x}{N})) f(x).$$

By choosing N sufficiently large, we can arrange that each g_t is arbitrarily close to f in the C^1-sense, and therefore a PD embedding. Then g_t is a PD isotopy from f to a map g_1, where $g_1 | V$ is linear on each simplex. Using Remark 6, we obtain the following:

Claim 7. *It suffices to solve Problem 5 in the special case where f is linear on each simplex.*

For $x \in K$. Choose a function $\chi : K \to \mathbb{R}_{>0}$ which is smooth on each simplex, nondecreasing on each ray from the origin, and satisfies the following conditions:

(1) The map χ is constant in a neighborhood of 0.

(2) The map χ is equal to 1 near ∂K.

(3) The map χ is given by $\chi(x) = \frac{s\epsilon}{|f(x)|}$ for $x \in s\,\partial K$ if $s \in [\frac{1}{4}, \frac{1}{2}]$, for some $\epsilon > 0$.

We define a PD isotopy f_t by the formula

$$f_t(x) = (1-t) f(x) + t\chi(x) f(x).$$

Then f_1 carries $s\,\partial K$ to the sphere of radius $s\epsilon$ for $s \in [\frac{1}{4}, \frac{1}{2}]$. Replacing f by f_1, applying an appropriate dilation to the target space $\mathbb{R}^{m+1}$, and invoking Remark 6, we are reduced to the following situation:

Claim 8. *It suffices to solve Problem 5 in the special case where $f(K)$ is the unit ball $B(1)$, and $f(tx) = tf(x)$ for $t \in [\frac{1}{2}, 1]$, $x \in \partial K$.*

The advantage of our present situation is that the image of ∂K now inherits a smooth structure from the map f. We will exploit this in the next lecture.