Flexibility (Lecture 16)

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Recall that our goal is to prove the following result:

Theorem 1. *Let M be a PL manifold. The above construction determines a homotopy equivalence from the simplicial set* Smooth(M) *of smooth structures on M to the simplicial set*

$$BO(m)^M \times_{BPL(m)^M} \{\chi\}$$

of liftings of χ. In particular, M admits a smoothing if and only if there exists a commutative diagram

$$\begin{array}{ccc} & & BO(m) \\ & \overset{L}{\nearrow} & \downarrow \\ M & \xrightarrow{\chi} & BPL(m). \end{array}$$

To prove Theorem 1, it will be convenient to formulate a more local version. For every open subset $U \subseteq M$, let Smooth(U) denote the simplicial set of smooth structures on U. The assignment $U \mapsto \text{Smooth}(U)$ defines a sheaf of simplicial sets on M. We can extend the definition of this sheaf to closed subpolyhedra $K \subseteq M$ by the formula $\text{Smooth}(K) = \varinjlim_{K \subseteq U} \text{Smooth}(U)$. We now have the following generalization of Theorem 1:

Theorem 2. *Let M be a PL manifold and $K \subseteq M$ a closed subpolyhedron. Then the above construction determines a homotopy equivalence from the simplicial set* Smooth(K) *of smooth structures on M to the simplicial set*

$$BO(m)^K \times_{BPL(m)^K} \{\chi|K\}$$

of liftings of $\chi|K$.

We observe that Theorem 2 is trivial in the case where K is a point: in this case, the map $\text{Smooth}(K) \rightarrow BO(m) \times_{BPL(m)} *$ is an isomorphism of simplicial sets.

In the statement of Theorem 2, the right hand side has a description in terms of sections of fibrations, and is thus under good homotopy-theoretic control. To prove Theorem 2, we will need a similar understanding of the left hand side. This is furnished by the following fact, which is the main objective of this lecture:

Proposition 3 (Flexibility)**.** *Let $K \subseteq K'$ be compact subpolyhedra of M. Then the restriction map* $\text{Smooth}(K') \rightarrow \text{Smooth}(K)$ *is a Kan fibration.*

Note that $\text{Smooth}(K') = \varinjlim_{K' \subseteq V} \text{Smooth}(V)$. Since a direct limit of Kan fibrations is a Kan fibration, it will suffice to prove that each of the maps $\text{Smooth}(V) \rightarrow \text{Smooth}(K)$ is a Kan fibration. Replacing M by V, we are reduced to proving the following:

Proposition 4. *Let K be a compact subpolyhedron of M. Then the restriction map* $\text{Smooth}(M) \rightarrow \text{Smooth}(K)$ *is a Kan fibration.*

We must show that every lifting problem of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Smooth}(M) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Smooth}(K) \end{array}$$

has a solution. The top map determines a PD homeomorphism $\Lambda^n_i \times M \to N$, where N is a smooth fiber bundle over Λ^n_i. Since the horn Λ^n_i is contractible, we can write $N = \Lambda^n_i \times N_0$, where N_0 is a smooth manifold. The bottom map determines an open subset U of $M \times \Delta^n$ containing $K \times \Delta^n$ and a PD homeomorphism $U \to W$, where W is a smooth fiber bundle over Δ^n whose restriction to Λ^n_i can be identified with an open subset of $\Lambda^n_i \times N_0$. Since Δ^n is trivial, we can write $W = W_0 \times \Delta^n$, where W_0 is a smooth manifold. Unwinding everything, we have the following data:

(1) A PD family $\{f_v : M \to N_0\}_{v \in \Lambda^n_i}$ of PD homeomorphisms.

(2) A PD homeomorphism $g : U \simeq \Delta^n \times W_0$, compatible with the projection to Δ^n.

(3) A smooth family of open embeddings $\{h_v : W_0 \to N_0\}_{v \in \Lambda^n_i}$ such that the following diagrams commute:

$$\begin{array}{ccc} U \times_{\Delta^n} \{v\} & \longrightarrow & M \\ \downarrow{\scriptstyle g_v} & & \downarrow{\scriptstyle f_v} \\ W_0 & \xrightarrow{h_v} & N_0. \end{array}$$

Let $B \subseteq N_0$ be a compact set containing the image of $K \times \Delta^n$ in its interior. Enlarging B, we may suppose that B is a smooth submanifold with boundary of N with codimention zero. Fix a point $0 \in \Lambda^n_i$. Using the parametrized isotopy extension theorem (in the smooth category), we can find a smooth family of diffeomorphisms $\{h'_v : M \to M\}_{v \in \Lambda^n_i}$ such that $(h'_v h_0)|B = h_v|B$. Replacing h_v by ${h'_v}^{-1} h_v$ and f_v by ${h'}_v^{-1} f_v$, we can assume that h_v is constant on the interior B. Replacing W_0 by the interior of B and shrinking U, we may assume that h_v is actually constant. We may therefore identify W_0 with an open subset of N_0.

To prove the existence of the desired extension, it will suffice to show that we can extend f_v to a PD family of PD homeomorphisms $\{f'_v : M \to N_0\}_{v \in \Delta^n}$, such that the families $\{f'_v\}$ and g agree in a neighborhood of K. Enlarging K, it will suffice to guarantee that we can arrange these maps to agree on K itself. Choose a PL homeomorphism $\Delta^n \simeq C \times [0,1]$, where $C = \Lambda^n_i$, and view $\{g_v\}_{v \in \Delta^n}$ as a two-parameter family $\{g_{c,t}\}_{c \in C, t \in [0,1]}$.

Note that f_v and g determine a polyhedral structure S on

$$(N_0 \times C) \coprod_{g(P) \times_{[0,1]} \{0\}} g(P)$$

where P is any closed subpolyhedron of U. Choose P to contain $K \times \Delta^n$. Our existence results for triangulations show that we can find a Whitehead compatible triangulation of $N_0 \times C \times [0,1]$ which is compatible with the projection to $C \times [0,1]$ and agrees with S near $N_0 \times C \times \{0\}$ and near $g(K \times C \times [0,1])$. Since the projection $\pi : N_0 \times C \times [0,1] \to C \times [0,1]$ is a fiber bundle in the smooth category, it is also a fiber bundle in the PL category, and can therefore be identified with $\pi^{-1}(C \times \{0\}) \times [0,1] \simeq M \times C \times [0,1]$. Using the parametrized isotopy extension theorem (in the PL category), we can adjust this identification so that it agrees with g on $K \times C \times [0,1]$. This provides the desired extension $\{f'_{c,t}\}_{c \in C, t \in [0,1]}$ of $\{f_c\}_{c \in C}$ and completes the proof.