Smoothings and Microbundles (Lecture 15)

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We now return to the problem of smoothing piecewise linear manifolds. Recall the diagram

$$\mathrm{Man}^m_{PL} \xrightarrow{\theta} \mathrm{Man}^m_{PD} \xrightarrow{\theta'} \mathrm{Man}^m_{\mathrm{sm}}$$

of Lecture 6. We have shown that θ' is a trivial Kan fibration, so that we can also regard Man^m_{PD} as a classifying space for smooth manifolds. Then we can regard θ as assigning to each smooth manifold an underlying PL manifold. The fiber of θ over a vertex of Man^m_{PL} corresponding to a PL manifold $M \subseteq \mathbb{R}^\infty$ can be viewed as a "space" of smooth structures on M. The following guarantees that these "spaces" of smooth structures are well-behaved:

Lemma 1. *The map θ is a Kan fibration.*

In fact, we will factor θ in two steps. Let $\mathrm{Man}^m_{PD'}$ denote the simplicial set whose k-simplices are fiber bundles of PL manifolds $E \rightarrow \Delta^k$ where $E \subseteq \Delta^k \times \mathbb{R}^\infty$, together with a Whitehead compatible smooth structure on E such that the map $E \rightarrow \Delta^k$ is a submersion (and therefore a fiber bundle) in the smooth category. This differs only slightly from our definition of Man^m_{PD}, in that we do not require an additional smooth embedding of E into $\Delta^k \times \mathbb{R}^\infty$. By general position arguments, this difference is immaterial: the map $\mathrm{Man}^m_{PD} \rightarrow \mathrm{Man}^m_{PD'}$ is a trivial Kan fibration. Consequently, it suffices to prove the following analogue of Lemma 1:

Lemma 2. *The map $\mathrm{Man}^m_{PD'} \rightarrow \mathrm{Man}^m_{PL}$ is a Kan fibration.*

Proof. We must show that we can solve lifting problems of the form

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathrm{Man}^m_{PD'} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Man}^m_{PL}. \end{array}$$

In more concrete terms: we are given a bundle of PL manifolds $K \subseteq \Delta^n \times \mathbb{R}^\infty$, and a PD homeomorphism of the subbundle $K_0 = K \times_{\Delta^n} \Lambda^n_i$ with a smooth fiber bundle $M_0 \rightarrow \Lambda^n_i$. We need to construct the following:

(1) A fiber bundle $M \rightarrow \Delta^n$ of smooth manifolds extending the given bundle $M_0 \rightarrow \Lambda^n_i$.

(2) A PD homeomorphism $K \rightarrow M$ which commutes with the projection to Δ^n.

To satisfy (1), we observe that Λ^n_i is trivial, so we can write M_0 as a product $\Lambda^n_i \times N$ for some smooth manifold N. We then define $M = \Delta^n \times N$. To construct (2), we observe that Δ^n is PL homeomorphic to $\Lambda^n_i \times \Delta^1$. We can lift this to a PL homeomorphism $K \simeq K_0 \times \Delta^1$. We now have a unique map $K \rightarrow \Delta^n \times N$ which commutes with the projection to Δ^n, and such that the map $K \rightarrow N$ is given by the composition

$$K \simeq K_0 \times \Delta^1 \rightarrow K_0 \rightarrow M_0 \simeq N \times \Lambda^n_i \rightarrow N.$$

It is easy to see that this map is a PD homeomorphism. □

Notation 3. Given a PL manifold M (which we implicitly assume to be given as a polyhedron in $\mathbb{R}^\infty$, so that it defines a vertex of Man^m_{PL}), we let $\mathrm{Smooth}(M)$ denote the fiber of the Kan fibration $\mathrm{Man}^m_{PD'} \to \mathrm{Man}^m_{PL}$ over M. The vertices of $\mathrm{Smooth}(M)$ are smooth structures on M which are Whitehead compatible with the given PL structure on M.

The theory of microbundles allows us to set up a local version of the same story. Namely, let $BPL(m)$ denote the classifying space (= simplicial set) for PL microbundles of rank m constructed in Lecture 12: an n-simplex of $BPL(m)$ is a microbundle $E \to \Delta^n$, where E is given as a subpolyhedron of $\Delta^n \times \mathbb{R}^\infty$. (The Kister-Mazur theorem, in its PL incarnation, allows us to identify this space with the classifying space of a simplicial group $PL(m)$).

Definition 4. Let E be a PL microbundle over a simplex Δ^n. Let us say that a *smoothing* of E is a smoothing of an open subset $U \subseteq E$ containing the zero section, so that the projection $U \to \Delta^n$ is submersive. We regard two smoothings as identical if they agree on a neighborhood of the zero section of E. Let $X_\bullet$ be the simplicial set whose n-simplices are pairs (σ, S), where σ is an n-simplex of $BPL(m)$ and S is a smoothing of the associated microbundle $E \to \Delta^n$. There is an evident forgetful map $f : X_\bullet \to BPL(m)$.

We can regard the map f as a "local version" of the Kan fibration $\theta : \mathrm{Man}^m_{PD} \to \mathrm{Man}^m_{PL}$. A slight modification of the proof of Lemma 2 shows that f is also a Kan fibration.

Lemma 5. *The vector bundle ζ over $X_\bullet$ constructed above is universal: that is, it exhibits $X_\bullet$ as a classifying space for vector bundles of rank m.*

Proof. By an argument which should be familiar from previous lectures, it will suffice to prove the following: given a map $\chi_0 : \partial\,\Delta^n \to X_\bullet$ and a vector bundle ζ' over Δ^n with an isomorphism $\alpha_0 : \zeta' | \partial\,\Delta^n \simeq \chi_0^* \zeta$, we can extend χ_0 to a map $\chi : \Delta^n \to X_\bullet$ and α to an isomorphism $\zeta' \simeq \chi^* \zeta$.

Since Δ^n is contractible, we can assume that ζ' is a trivial bundle of rank m. The map χ_0 classifies a PL microbundle $E_0 \to \partial\,\Delta^n$ (together with an embedding $E_0 \hookrightarrow \partial\,\Delta^n \times \mathbb{R}^\infty$), and a smoothing S of a neighborhood U_0 of the zero section of E_0. The map α_0 gives a trivialization of vertical tangent space to U_0 along the zero section. As we have seen, this is equivalent to trivializing U_0 as a smooth microbundle. We may therefore assume, after shrinking U, that $U_0 \simeq \partial\,\Delta^n \times \mathbb{R}^m$ as a smooth fiber bundle over $\partial\,\Delta^n$.

We wish to show that we can extend E_0 to a PL microbundle $E \to \Delta^n$ (which we can then embed in $\Delta^n \times \mathbb{R}^\infty$ using general position arguments) and U_0 to an open subset $U \subseteq E$ containing the zero section, equipped with a PD homeomorphism $U \to \mathbb{R}^m \times \Delta^n$. To construct this, choose a finite polyhedral neighborhood V of $\partial\,\Delta^n$ in Δ^n for which there exists a retraction $r : V \to \Delta^n$. Let V_0 denote the interior of V, and let r_0 be the restriction of r to V_0, and let $\partial\,V = V - V_0$. Let E denote the pushout

$$(r_0^* E_0) \coprod_{r_0^* U_0} (\Delta^n \times \mathbb{R}^\infty)$$

Over V, this set is equipped with a natural polyhedral structure by identifying it with an open subset of $r^* E_0$. In particular, we get a PL structure on $E \times_{\Delta^n} \partial\,V \simeq (\partial\,V) \times \mathbb{R}^m$ which is Whitehead compatible with the smooth structure on $\mathbb{R}^m$. We now simply extend this to a triangulation of the smooth fiber bundle

$$E \times_{\Delta^n} (\Delta^n - V_0) \simeq \mathbb{R}^m \times (\Delta^n - V_0) \to \Delta^n - V_0$$

to obtain the desired PL microbundle E. □

Since $X_\bullet$ is classifying space for vector bundles, we will denote it by $BO(m)$: it is homotopy equivalent to any other model for the classifying space $BO(m)$ (for example, one constructed using the singular complex of the topological group $O(m)$). By construction, we have a Kan fibration $\theta_0 : BO(m) \to BPL(m)$. Informally, we think of this as coming from a group homomorphism $O(n) \to PL(n)$. (In fact, we do have an evident morphism from $O(n)$ to $PL(n)$ as discrete groups: every orthogonal transformation of $\mathbb{R}^n$ is in particular a piecewise linear homeomorphism.) The fiber of f is often denoted $PL(n)/O(n)$; it can be thought of as the space of all smoothings of the PL manifold $\mathbb{R}^n$.

Let us adopt the following convention: if M is a polyhedron and $Y_\bullet$ is a simplicial set, then a *map* from M into $Y_\bullet$ means a map of simplicial sets from the PL singular complex $\mathrm{Sing}_\bullet^{PL} M$ into $Y_\bullet$. The collection of all such maps can itself be organized into a simplicial set which we will denote by $Y_\bullet^M$.

If M is a PL manifold of dimension m, then there is a natural map $\chi : M \to BPL(m)$: namely, it assigns to each n-simplex $\sigma : \Delta^n \to M$ the product $M \times \Delta^n$, regarded as a PL microbundle over Δ^n with the section supplied by σ. Any smoothing of M determines a smoothing of this PL microbundle: in other words, it allows us to produce a lifting

$$\begin{array}{ccc} & & BO(m) \\ & \nearrow & \downarrow \\ M & \longrightarrow & BPL(m). \end{array}$$

Our goal in the next few lectures is to prove the converse. More precisely, we will show the following:

Theorem 6. *Let M be a PL manifold. The above construction determines a homotopy equivalence from the simplicial set* Smooth(M) *of smooth structures on M to the simplicial set*

$$BO(m)^M \times_{BPL(m)^M} \{\chi\}$$

of liftings of χ. In particular, M admits a smoothing if and only if there exists a commutative diagram

$$\begin{array}{ccc} & & BO(m) \\ & \overset{L}{\nearrow} & \downarrow \\ M & \xrightarrow{\chi} & BPL(m). \end{array}$$

The virtue of this result is that it reduces the classification of smooth structures on M to a problem of homotopy theory. The existence of the arrow L can in principle be attacked by methods of obstruction theory. Namely, consider the fiber of the Kan fibration $BO(m) -> BPL(m)$, which we will suggestively denote by $PL(m)/O(m)$ (it can be thought of as the space of all smooth structures on the trivial PL microbundle $\mathbb{R}^m \to *$). Obstruction theory tells us that L will exist provided that a sequence of cohomology classes $\mathrm{H}^k(M; \pi_{k-1} PL(m)/O(m))$ vanish Similarly, the uniqueness of L can be studied by computing cohomology groups of the form $\mathrm{H}^k(M; \pi_k PL(m)/O(m))$. In particular, if the homotopy groups of $PL(m)/O(m)$ vanish, then M admits an essentially unique smooth structure. This is what happens for $m \leq 3$, as we will see later.