Embeddings vs. Homeomorphisms (Lecture 13)

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Our goal in this lecture is to carry out the main step in the proof of the Kister-Mazur theorem describing the relationship between microbundles and $\mathbb{R}^n$-bundles. Namely, we will prove the following:

Theorem 1. *Let* $\mathrm{Emb}(\mathbb{R}^n)$ *denote the simplicial set of open embeddings from* $\mathbb{R}^n$ *to itself (so a* k*-simplex of* $\mathrm{Emb}(\mathbb{R}^n)$ *is an open embedding* $j : \mathbb{R}^n \times \Delta^k \to \mathbb{R}^n \times \Delta^k$ *which commutes with the projection to* Δ^k*), and let* $\mathrm{Homeo}(\mathbb{R}^n) \subseteq \mathrm{Emb}(\mathbb{R}^n)$ *denote the simplicial subset of homeomorphisms from* $\mathbb{R}^n$ *to itself (so that a* k*-simplex of* $\mathrm{Homeo}(\mathbb{R}^n)$ *is a* k*-simplex of* $\mathrm{Emb}(\mathbb{R}^n)$ *for which the map* j *is a homeomorphism). Then the inclusion* $i : \mathrm{Homeo}(\mathbb{R}^n) \subseteq \mathrm{Emb}(\mathbb{R}^n)$ *is a homotopy equivalence of Kan complexes.*

Remark 2. We can also define topological spaces parametrizing homeomorphisms or open embeddings from $\mathbb{R}^n$ to itself: Theorem 1 is equivalent to the assertion that the inclusion between these topological spaces is a weak homotopy equivalence.

Remark 3. We can also define simplicial sets which parametrize PL embeddings and PL homeomorphisms from $\mathbb{R}^n$ to itself. Theorem 1 continues to hold in this case, using essentially the same proof that we will give below.

The main step in the proof of Theorem 1 is to establish that i is a surjection on π_0. In other words, every open embedding $f : \mathbb{R}^n \to \mathbb{R}^n$ is isotopic to a homeomorphism of $\mathbb{R}^n$ with itself. In fact, we will prove something more precise:

Proposition 4. *Let* f *be an open embedding from* $\mathbb{R}^n$ *to itself. Then there exists an isotopy* F_t *from* $f = F_0$ *to a homeomorphism* $f = F_1$*. Moreover, this isotopy is can be chosen to be constant on the unit ball* $B(1)$ *of* $\mathbb{R}^n$*.*

Notation 5. For every positive real number r, let $B(r) = \{x \in \mathbb{R}^n : |x| < r\}$ be the open ball of radius r around the origin (in giving the PL version of this proof, it is convenient to replace $B(r)$ by an open cube).

Here is the rough idea of the proof. The obstruction to an open embedding being a homeomorphism is that it might not be surjective. Our objective, therefore, is to use an isotopy to modify f so that its image becomes larger and larger. More precisely, we will construct a sequence of open embeddings

$$f^1, f^2, \ldots, : \mathbb{R}^n \to \mathbb{R}^n$$

and a sequence of isotopies $h^i : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ so that the following conditions are satisfied:

(1) The map $f^1 = f$.

(2) For each i, the map h^i is an isotopy from $f^i = h^i_0$ to $f^{i+1} = h^i_1$, which is constant on the open ball $B(i)$.

(3) For $i > 1$, we have $B(i) \subseteq f^i B(i)$.

Assuming that we can meet these requirements, we can define a homeomorphism $f' : \mathbb{R}^n \to \mathbb{R}^n$ by the formula $f'(x) = f^i(x)$ for any $i \geq |x|$. We get an isotopy from f to f' by concatenating the isotopies h^1, h^2, and so forth (this concatenation is well-defined since almost all of the isotopies h^i are constant on any given compact subset of $\mathbb{R}^n$).

To begin, we may assume without loss of generality that $f(0) = 0$ (otherwise, we can reduce to this case by conjugating by a relevant translation). Since f is an open embedding, the image $fB(1)$ contains an open ball $B(\epsilon)$ for some real number $\epsilon > 0$. Since f is continuous, there exists a positive real number $\delta < 1$ such that $f(B(\delta)) \subseteq B(\frac{\epsilon}{2})$.

To construct our isotopies h^i, we will need the following basic building blocks:

Notation 6. For every pair of real numbers $r < s$, we fix an isotopy $H(r,s)_t : \mathbb{R}^n \to \mathbb{R}^n$ from $\mathrm{id}_{\mathbb{R}^n}$ to $H(r,s)_1$ with the following properties:

- (i) The isotopy $H(r,s)_t$ is trivial on $B(\frac{r}{2})$ and supported in a compact subset of $B(s+1)$.
- (ii) The map $H(r,s)_1$ restricts to a homeomorphism $B(r)$ to $B(s)$.

We now proceed with the construction of the sequence $\{f^i\}$. Assume that f^i has already been constructed. We wish to construct an isotopy h^i from f^i to another map f^{i+1}, which is constant on $B(i)$. First, we define a homeomorhism c (for "contraction") from $\mathbb{R}^n$ to itself as follows:

$$c(x) = \begin{cases} x & \text{if } x \notin f^i(\mathbb{R}^n) \\ f^i(H(\delta, i)_1^{-1}(y)) & \text{if } x = f^i(y). \end{cases}$$

Since $f^i = f$ on $B(1)$ and f carries $B(\delta)$ into $B(\frac{\epsilon}{2})$, we deduce that $c(f^i(x)) \in B(\frac{\epsilon}{2})$ if $x \in B(i)$. Note that c is the identity outside a compact set, which we can take to be contained in $B(N_i)$ for some $N_i \gg i+1$.

We now define h^i_t by the formula

$$h^i_t = c^{-1} \circ H(\epsilon, N_i)_t \circ c \circ f^i.$$

It is clear that h^i_t is an isotopy from $f^i = h^i_0$ to another map $f^{i+1} = h^i_1$. Moreover, since $H(\epsilon, N_i)_t$ is the identity on $B(\frac{\epsilon}{2})$ and $c \circ f^i$ carries $B(i)$ into $B(\frac{\epsilon}{2})$, we deduce that h^i_t is constant on $B(i)$. It remains only to verify that $f^{i+1}B(i+1)$ contains $B(i+1)$. In fact, we claim that $f^{i+1}B(i+1)$ contains $B(N_i)$. Since c is supported in $B(N_i)$, it suffices to show that $(cf^{i+1})B(i+1) = (H(\epsilon, N_i)_1 \circ c \circ f^i)B(i)$ contains $B(N_i)$. For this, it suffices to show that $(c \circ f^i)B(i)$ contains $B(\epsilon) \subseteq fB(1) \subseteq f^iB(i+1)$. This is clear, since $H(\delta, i)_1$ induces a homeomorphism of $B(i+1)$ with itself. This completes the proof of Proposition 4.

Remark 7. In the above construction, each of the isotopies h^i is obtained by composing f^i with a 1-parameter family $c^{-1} \circ H(\epsilon, N_i)_t \circ c$ of homeomorphisms from $\mathbb{R}^n$ to itself. It follows that if the original map f is already a homeomorphism, then the isotopy F_t that we construct will be a path through the space of homeomorphisms.

Suppose now that we are given not a single open embedding $f : \mathbb{R}^n \to \mathbb{R}^n$, but a family of open embeddings $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta$ (compatible with the projection to Δ), where Δ is some parameter space. We might try to apply the above construction to each of the induced maps $\{f_v : \mathbb{R}^n \to \mathbb{R}^n\}_{v \in \Delta}$ to produce a family of isotopies $\{F_{v,t} : \mathbb{R}^n \to \mathbb{R}^n\}_{(v,t) \in \Delta \times [0,1]}$. We must be careful, since our construction depended on several choices. First of all, we needed to choose ϵ such that $f_v B(1)$ contains the open ball $B(\epsilon)$. We note that $f(B(1) \times \Delta)$ is an open neighborhood of $\{0\} \times \Delta$ in $\mathbb{R}^n \times \Delta$, which will contain some product neighborhood $B(\epsilon) \times \Delta$ provided that Δ is compact. We also needed to choose a constant δ such that $f_v B(\delta) \subseteq B(\frac{\epsilon}{2})$. Again, if Δ is compact, then a sufficiently small real number δ will work for all f_v's simultaneously. Finally, to constuct each h^i_v we needed to choose $N_i \gg i+1$, so that the relevant contraction c_v has compact support in $B(N_i)$. The support of c_v is contained in $f^i_v B(i+1)$. If Δ is compact, the image $f^i(B(i+1) \times \Delta)$ will be contained in a compact subset of $\mathbb{R}^n \times \Delta$, which is in turn contained in $B(N_i) \times \Delta$ for sufficiently large N_i. Consequently, we get the following more refined version of Proposition 4:

Proposition 8. *Let Δ be a compact topological space (for example, a simplex), and suppose we are given an open embedding $f : \mathbb{R}^n \times \Delta \to \mathbb{R}^n \times \Delta$ which is compatible with the projection to Δ. Then there exists an isotopy $F : \mathbb{R}^n \times \Delta \times [0,1] \to \mathbb{R}^n \times \Delta \times [0,1]$ with the following properties:*

(1) *The map F_0 coincides with f.*

(2) *The map F_1 is a homeomorphism.*

(3) *The isotopy F is constant along $B(1) \times \Delta$.*

(4) *If f_v is already a homeomorphism for some $v \in \Delta$, then the isotopy $F_v : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \times [0,1]$ consists of homeomorphisms.*

We can now prove Theorem 1. The proof is based on the following criterion for detecting homotopy equivalences:

Proposition 9. *Let $i : K \subseteq K'$ be an inclusion of Kan complexes. Then i is a homotopy equivalence if and only if the following condition is satisfied:*

$(*)$ *For every n-simplex σ of K' whose boundary belongs to K, there exists a homotopy $h : \Delta^n \times \Delta^1 \to K'$ such that $h|\Delta^n \times \{0\} = \sigma$, $h|\Delta^n \times \{1\}$ factors through K, and $h|\,\partial\,\Delta^n \times \Delta^1$ factors through K.*

Roughly speaking, the simplex σ is a typical representative of a class in π_{n-1} of the homotopy fiber of the inclusion $K \to K'$, and condition $(*)$ guarantees that any such class is trivial.

Theorem 1 follows immediately from Proposition 9 and Proposition 8.

In the next lecture, we will discuss the consequences of Theorem 1 for the classification of microbundles.