Microbundles (Lecture 11)

February 27, 2009

In this lecture, we will continue our study of microbundles. Recall that a microbundle over X is a map $p : E \to X$ equipped with a section $s : X \to E$. We will sometimes abuse terminology and simply refer to $p : E \to X$ or just the space E as a microbundle.

Remark 1. Let $E \to X$ be a topological (PL, smooth) microbundle, and let $f : X' \to X$ be a continuous (PL, smooth) map. Then the pullback $X' \times_X E \to X'$ is a microbundle over X', which we will denote by f^*E.

Our goal for this lecture is to prove the following:

Theorem 2. *Let $f, f' : X \to X'$ be a pair of continuous maps between topological spaces, and let E be a microbundle over X'. If X is paracompact and the maps f and f' are homotopic, then the microbundles f^*E and ${f'}^*E$ are equivalent.*

Remark 3. We have stated Theorem 2 in the topological setting, but it has obvious analogues in the smooth and PL settings. These can be proven using the same arguments given below; we will stick to the topological case just to save words.

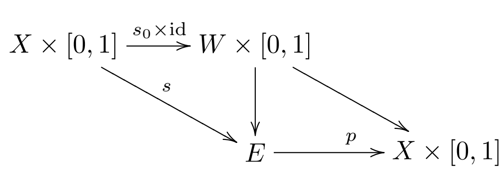
Corollary 4. *We say that a microbundle $E \to X$ is trivial if it is equivalent to a product $\mathbb{R}^n \times X$. If X is paracompact and contractible, then every microbundle over X is trivial.*

Proof. The identity map id_X is homotopic to a constant map $c : X \to X$ taking values at some point $x \in X$, so any microbundle E on X is equivalent to $c^*E = X \times E_x$. Since E is a microbundle, the fiber E_x has an open subset homeomorphic to $\mathbb{R}^n$ (containing $s(x)$). □

We now turn to the proof of Theorem 2. A homotopy between a pair of maps $f, f' : X \to X'$ is a map $h : X \times [0,1] \to X'$. To prove that $f^* \simeq {f'}^*E$, it will suffice to show that $h^*E \simeq \pi^*E_0$, where E_0 is a microbundle on X and $\pi : X \times [0,1] \to X$ is the projection map. We may therefore reformulate Theorem 2 as follows:

Proposition 5. *Let X be a paracompact space and let $E \to X \times [0,1]$ be a microbundle. Then there exists an equivalence of E with $E_0 \times [0,1]$, where E_0 denotes the fiber $E \times_{[0,1]} \{0\}$.*

In other words, there exists an open subset W of E_0 (containing the image of the section $s_0 : X \to E_0$) and an open embedding $W \times [0,1] \to E$ such that the diagram



is commutative.

We first treat the case where X is a point. In this case, E is a microbundle over the interval $[0,1]$ and we wish to prove that E is trivial. For each $x \in [0,1]$, there exists an open subset of E homeomorphic to a product $U \times V$, where V is a neighborhood of x in $[0,1]$ and $U \times V$ contains the image of the section s. Since $[0,1]$ is compact, we can cover $[0,1]$ by finitely many of the neighborhoods V. It follows that there exists an integer $N \gg 0$ and open embeddings

$$h(i) : U_i \times [\frac{i-1}{N}, \frac{i}{N}] \hookrightarrow E$$

for $1 \leq i \leq N$, where U_i is a space containing a base point $*$ and $h(i)$ carries $(*, t)$ to $s(t)$ for $t \in [\frac{i-1}{N}, \frac{i}{N}]$. We can think of each $h(i)$ as a family of open embeddings $h(i)_t : U_i \rightarrow E_t$, parametrized by $t \in [\frac{i-1}{N}, \frac{i}{N}]$. Using decreasing induction on $i < N$, we can assume (after shrinking U_i) that the map $h(i)_{\frac{i}{N}}(U_i) \subseteq h(i+1)_{\frac{i}{N}}(U_{i+1})$. We can then define a single map $f : U_0 \times [0,1] \rightarrow E$ by the following formula:

$$g(u,t) = h(i)_t h(i)^{-1}_{\frac{i-1}{N}} h(i-1)_{\frac{i-1}{N}} h(i-2)^{-1}_{\frac{i-2}{N}} \ldots h(1)_{\frac{1}{N}}(u)$$

where $\frac{i-1}{N} \leq t \leq \frac{i}{N}$. It is easy to see that g determines a trivialization of the microbundle E.

Now consider a general topological space X, and let $x \in X$ be a point. We can repeat the above argument to find an integer N and a finite sequence of open embeddings

$$h(i) : U_{i,x} \times [\frac{i-1}{N}, \frac{i}{N}] \times V_i \hookrightarrow E$$

where V_i is a sequence of open neighborhoods of x in X. Replacing each V_i by the intersection $V_x = \bigcap V_i$, we can assume that all of the open sets V_i are the same. After shrinking the open subsets $U_{i,x}$ as above, we can again define an open embedding $g_x : U_{0,x} \times V_x \times [0,1] \hookrightarrow E$ by setting

$$g_x(u,v,t) = h(i)_t h(i)^{-1}_{\frac{i-1}{N}} h(i-1)_{\frac{i-1}{N}} h(i-2)^{-1}_{\frac{i-2}{N}} \ldots h(1)_{\frac{1}{N}}(u,v)$$

where $\frac{i-1}{N} \leq t \leq \frac{i}{N}$. This open embedding determines a trivialization of the microbundle E on a neighborhood $[0,1] \times V_x$ of $[0,1] \times \{x\}$.

Since X is paracompact, we can choose a locally finite open covering $\{V_\alpha\}_{\alpha \in A}$ refining the covering $\{V_x\}_{x \in X}$ of X. For each $\alpha \in A$, choose a point $x \in X$ such that $V_\alpha \subseteq V_x$, let $W_\alpha = U_{0,x} \times V_x$ (which we identify with an open subset of E_0), let let $g_\alpha : W_\alpha \times [0,1] \rightarrow E$ be the restriction of g_x.

Each g_α determines an equivalence of E with the microbundle $\pi^* E_0$ over the open subset $W_\alpha \times [0,1] \subseteq X \times [0,1]$. We would like to "average" these equivalences to obtain a new equivalence G over all of $X \times [0,1]$. To this end, we choose choose a linear ordering on the set A and a partition of unity $\{\psi_\alpha : X \rightarrow [0,1]\}_{\alpha \in A}$ subordinate to the covering V_α. We attempt to define a map $G : E_0 \times [0,1] \rightarrow E$ as follows. Fix a point $e \in E_0$ lying over a point $x \in X$. Since the covering $\{V_\alpha\}$ is locally finite, x is contained V_α for only a finite number of indices $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ of A. For each $t \in [0,1]$, choose an index i such that

$$\psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_{i-1}}(x) \leq t \leq \psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_i}(x)$$

and set

$$G(e,t) = g_{\alpha_i,t} g^{-1}_{\alpha_i, \psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_{i-1}}(x)} g_{\alpha_{i-1}, \psi_{\alpha_1}(x) + \ldots + \psi_{\alpha_{i-1}}(x)} \cdots g_{\alpha_1, \psi_{\alpha_1}(x)}(e).$$

The map G is not everywhere defined, since the functions $g^{-1}_{\alpha,t}$ are defined only on open subsets of $E_t \times_X V_\alpha$ and the functions $g_{\alpha,t}$ are defined only on open subsets of $E_0 \times_X V_\alpha$. However, the composition is well-defined on the subset $s_0(X) \times [0,1] \subseteq E_0 \times [0,1]$, and therefore on an open neighborhood of this subset. Since $[0,1]$ is compact, we can choose this open neighborhood to be of the form $W \times [0,1]$, where W is an open subset of E_0 containing $s_0(X)$. Then $G : W \times [0,1] \rightarrow E$ is an open embedding which provides the desired equivalence $E \simeq \pi^* E_0$.