In the early 1830s, William Rowan Hamilton provided an elegant reformulation of classical Newtonian physics. To a modern mathematician, Hamilton’s formulation suggests that the correct setting for classical mechanics is a symplectic manifold: a smooth phase space $M$ equipped with an additional structure $\omega$ called a ‘symplectic form’ governing the time-evolution of the system. If the system has symmetry, as it usually will, there is a group $G$ acting on $M$ by diffeomorphisms preserving $\omega$.

Almost one hundred years later, John Von Neumann provided the first rigorous formulation of quantum mechanics. He convincingly demonstrated that the correct mathematical setting for quantum mechanics is a complex Hilbert space: a (usually infinite-dimensional) vector space $V$ of probability distributions equipped with an inner product $\langle \cdot, \cdot \rangle$. In this world, a symmetry is a linear action of $G$ on $V$ preserving $\langle x, y \rangle$—in other words, a unitary representation.

The Orbit Method. Classical mechanics and quantum mechanics are radically different theories, but in a sense they provide descriptions of the same underlying reality. With enough imagination, we should be able to translate from one theory to the other. This correspondence, if it exists, should preserve symmetries, if there are any. In other words, what we want is a mapping:

$\{\text{Symplectic Manifolds with } G\text{-Symmetry}\} \rightarrow \{\text{Unitary Reps of } G\}$

In the 1960s, Alexandre Kirillov and Bertram Kostant discovered a correspondence of this form under some strong additional assumptions ([7], [8]). They observed that there is a rich supply of symplectic manifolds arising from the ‘internal’ geometry of $G$. The group $G$ acts naturally on the (real) dual of its Lie algebra $\mathfrak{g}_0^*$. The orbits of this action (called the ‘co-adjoint orbits’) are naturally symplectic manifolds with $G$-symmetry. Kirillov proved

**Theorem 1** (Kirillov, [7]). *If $G$ is simply connected and nilpotent, there is a bijective correspondence

$\{\text{Co-adjoint } G\text{-Orbits}\} \rightarrow \{\text{Irreducible Unitary Reps of } G\}$

This is not far from what we wanted: by an easy (but remarkable) theorem of Kostant ([8]), essentially every symplectic manifold with a nice, transitive $G$-action is a co-adjoint orbit for $G$. The transitivity assumption imposes a ‘smallness’ condition on $M$, which shows up as an irreducibility condition on $V$.

Theorem 1 provides an elegant classification of the unitary representations of a nilpotent Lie group. For this class of groups, it is the best classification available. It is called the ‘method of co-adjoint orbits’, or the ‘orbit method’, for short.

If we replace $G$ with a (noncompact) reductive group—like $GL_n(\mathbb{R})$, for example—almost every aspect of this correspondence breaks down. The techniques developed by Kostant and Kirillov no longer work for every co-adjoint orbit. When they do, often several representations are produced. Many representations do not arise through these methods from any co-adjoint orbit, and some arise from many. The perfect one-to-one correspondence of Kostant and Kirillov becomes a heap of imperfect heuristics. Nonetheless, these heuristics are among the best tools that we have.

Nilpotent Orbits and Unipotent Representations. The most interesting and consequential failure of the Orbit Method for reductive groups concerns the nilpotent co-adjoint orbits. Roughly speaking, the problem is this: none of the usual methods for producing unitary representations can be systematically applied to these orbits. And yet there is a small set of representations to which they ought to correspond.

In [17], Vogan calls these representations ‘unipotent’ and offers a provisional definition. Over the past several decades, these representations have been the subject of a large body of work. They are in a sense

---

1The Lie algebra $\mathfrak{g}_0$ of $G$ admits a non-degenerate, symmetric, bilinear form, which provides an identification $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$. An element $\lambda \in \mathfrak{g}_0^*$ is nilpotent if it corresponds under this identification to a nilpotent element $e \in \mathfrak{g}_0$. If $\mathfrak{g}_0 \subset \mathfrak{g}_n(\mathbb{C})$, this means simply that $e$ is a nilpotent matrix.
the most important representations of a real reductive group: empirical evidence suggests that every unitary representation can be built up in a suitable sense from unipotent representations, so an understanding of these representations is fundamental to the unitary representation theory of real reductive groups.

**Towards a General Theory of Unipotent Representations.** Let $G$ be a real reductive group and let $L \subset G$ be a Levi subgroup. There is a general procedure, called induction, for producing nilpotent orbits for $G$ from nilpotent orbits for $L$. To understand unipotent representations in general, I believe one must

1. understand the unipotent representations attached to *non-induced* nilpotent orbits, and
2. develop an appropriate notion of *induction* for unipotent representations

Towards the first goal, I have developed a theory of microlocalization for Harish-Chandra modules, inspired by the work of Losev on $W$-algebras and primitive ideals. In [12], I apply this theory to the unipotent representations attached to non-induced orbits to deduce an explicit formula for their $K$-types, proving (in a large family of cases) an old conjecture of Vogan (see [15], Conjecture 12.1).

Towards the second goal, I have introduced a candidate notion of induction for unipotent representations, generalizing the usual notion of induction due to Zuckerman and Vogan. I have proved that this new kind of induction behaves nicely with respect to the orbit method and the induction of nilpotent orbits. The representations thus produced are analogous to Lusztig’s $R_{P,1}$ characters in the representation theory of finite Chevalley groups. They are reducible in general, but related to the (irreducible) unipotent representations by an upper triangular matrix with $\pm 1$’s along the diagonal.

In the special case of the principal nilpotent orbit (which is always induced from the 0-orbit), I have developed a complete and explicit description of the corresponding representations. Roughly: every such representation is cohomologically induced from a spherical principal series representation of a split Levi subgroup.

The theory of unipotent representations suggests the existence of certain distinguished sheaves on the nilpotent cone. In recent joint work with James Tao (MIT), I have developed a collection of geometric tools for understanding these sheaves. In particular, our work provides a classification of all (graded, equivariant) coherent sheaves on normal varieties associated with induced nilpotent orbits. This classification provides important clues as to what these distinguished sheaves should look like, and has led us to a conjectural definition of ‘unipotent sheaves.’

**Details and Future Research**

Let $G$ be a real reductive group (one can assume, for example, that $G$ is the real points of a connected reductive algebraic group). Let $K \subset G$ be a maximal compact subgroup corresponding to a Cartan involution $\theta$. Write $G$ and $K$ for the complexifications of $G$ and $K$, respectively, $\mathfrak{g}$ for the Lie algebra of $G$, $\mathfrak{p}$ for the $-1$ eigenspace of $\theta$ on $\mathfrak{g}$, and $\mathcal{N} \subset \mathfrak{g}$ for the nilpotent cone. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and form the Langlands dual group $G^\vee$. By construction, $\mathfrak{g}^\vee$ contains a distinguished Cartan subalgebra $\mathfrak{h}^\vee$ which is naturally identified with $\mathfrak{h}^*$. If we write $N^\vee$ for the cone of nilpotent elements in $\mathfrak{g}^\vee$, there is an order-reversing map

$$\psi : N^\vee / G^\vee \to N / G$$

first defined by Spaltenstein ([14]). If $O^\vee \subset N^\vee$ is a $G^\vee$-orbit on $N^\vee$, Jacobson-Morozov theory hands us an element $h \in \mathfrak{h}^\vee$, unique up to conjugation by the Weyl group. The element $\frac{1}{2}h$ determines an infinitesimal character $\chi_{O^\vee}$ for $g$ by the Harish-Chandra isomorphism.

If $X$ is a finite-length $(\mathfrak{g},K)$-module, there is an associated class $\text{gr}(X)$ in the Grothendieck group of $K$-equivariant coherent sheaves on $N \cap \mathfrak{p}$. It is defined by taking the associated graded of $X$ with respect to a good filtration. This class has well-defined (set-theoretic) support $AV(X) \subset N \cap \mathfrak{p}$ which we call the associated variety of $X$. If $I$ is the annihilator of $X$ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, one can also define $AV(I) := V(\text{gr}(I)) \subset N$. These two notions are related: $AV(I)$ is the $G$-saturation of $AV(X)$.

The Arthur Conjectures ([2],[3]) suggest the following definition

**Definition 1.** Let $O$ be a $G$-orbit on $N$. An irreducible $(\mathfrak{g},K)$-module $X$ is a unipotent representation attached to $O$ if there is a nilpotent $G^\vee$-orbit $O^\vee \subset N^\vee$ with $\psi(O^\vee) = O$ and

1. $X$ has infinitesimal character $\chi_{O^\vee}$
2. $AV(I) = \overline{O}$
Unipotents Attached to Orbits of Small Boundary. In the setting described above, $AV(X)$ decomposes into finitely-many $K$-orbits. If we write $O_1, \ldots, O_n$ for the open $K$-orbits on $AV(X)$, then the restriction $gr(X)|_{O_i}$ defines, for each $i$, a virtual $K$-equivariant vector bundle $V_i \to O_i$. If $X$ is unipotent (in the sense of Definition 1, for example), then Vogan proves in [15] that these vector bundles have a very special form (very roughly speaking, they are equivariant local systems). In the same paper, he formulates the following conjecture

**Conjecture 1** (Vogan,[15]). Suppose $O$ is a $G$-orbit on $N$ satisfying
\[ \text{codim}(\partial O, \mathcal{O}) \geq 4 \]
and let $X$ be a unipotent representation attached to $O$ in the sense of Definition 1. Then $AV(X) = \mathcal{O}_1$, $V_1$ is irreducible, and
\[ X \cong_K \Gamma(O_1, V_1) \]

It is worth noting that the codimension condition in Conjecture 1 is closely related to the condition of being a non-induced orbit (see, e.g. [10]).

In [12], I prove:

**Theorem 2** (Mason-Brown, [12]). Suppose $G$ is complex, and suppose $O$ is a $G$-orbit on $N$ satisfying
\[ \text{codim}(\partial O, \mathcal{O}) \geq 6 \]
then $AV(X) = \mathcal{O}_1$, $V_1$ is irreducible, and
\[ X \cong_K \Gamma(O_1, V_1) \]

In unpublished work, I prove an analogous statement for $G = Sp(2n, \mathbb{R})$. To prove these results, I construct an endo-functor $\Phi_O$ on the category of finitely-generated $(\mathfrak{g}, K)$-modules, adapting a beautiful construction of Losev ([9]). Heuristically, $\Phi_O$ ‘microlocalizes over $O$.’ If $X$ is a unipotent representation attached to $O$, then one can show that
\[ X \cong \Phi_O X \]

Theorem 2 is deduced from this isomorphism, together with a vanishing theorem for nilpotent $K$-orbits which I also prove in [12].

The restrictions on $G$ can likely be relaxed (and I intend to relax them in future work). Relaxing the bound on the codimension of $\partial O$ will require some genuinely new ideas.

Unipotents Attached to Principal Orbits. Let $O^p \subset N$ be the principal nilpotent $G$-orbit. Write $Unip(O^p)$ for the set of (isomorphism classes of) unipotent representations attached to $O^p$ in the sense of Definition 1. Using the Langlands classification, I have developed a satisfying description of $Unip(O^p)$. My main result is the following

**Theorem 3** (Mason-Brown, [13]). There is a bijective correspondence between $Unip(O^p)$ and $K$-conjugacy classes of triples $(L, \mathfrak{q}, \chi)$ consisting of a split Levi subgroup $L \subset G$, a $\theta$-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, and a character $\chi$ of $L$ satisfying $d\chi = -\rho(\mathfrak{u})$ subject to the condition that $N_\chi \cap \mathfrak{p} + \mathfrak{u} \cap \mathfrak{p}$ contains a principal nilpotent element.

The representation $X(L, \mathfrak{q}, \chi)$ corresponding to the triple $(L, \mathfrak{q}, \chi)$ is given by
\[ X(L, \mathfrak{q}, \chi) = CohInd_{\mathfrak{q}, L}^{\mathfrak{b}, K}(\mu_{\mathfrak{q}} L, \chi) \otimes S_0(L) \]
where $CohInd$ is the functor of cohomological induction and $S_0(L)$ is the spherical principal series representation of $L$ of infinitesimal character 0.

One appealing feature of Theorem 3 is that $K$-structure and associated variety of $X(L, \mathfrak{q}, \chi)$ can be easily read off of equation 1. The data $(L, \mathfrak{q}, \chi)$ determines a closed subvariety $AV_\mathfrak{q} \subset N \cap \mathfrak{p}$, a proper surjection $\mu : N_\mathfrak{q} \to AV_\mathfrak{q}$ (resembling the Springer resolution of $N$), and a coherent sheaf $\mathcal{E}_{\mathfrak{q}, \chi}$ on $N_\mathfrak{q}$. In [13], I show that
\[ [gr(X)] = \sum_i (-1)^i [R^i \mu_* \mathcal{E}_{\mathfrak{q}, \chi}] \]
This leads to a Blattner-type formula for the $K$-types of $X$. I would like to replicate this result for non-principal orbits.
Unipotents Attached to Induced Orbits. Let \( q = l \oplus u \) be a parabolic subalgebra of \( g \). Then the quotient group

\[
R := \frac{Q \cap K}{U \cap K} \subset L
\]

acts on \( I \) via the isomorphism \( I \cong q/u \). Since \( q \) is not assumed to have any compatibility with \( \theta \), the subgroup \( R \subset L \) can be quite strange. In general, it is neither reductive nor parabolic, but a hybrid of the two. In [13], I consider a left-exact functor

\[
I_\mathbb{Q}^q : M(I,R) \to M(g,K)
\]

from the category of finite-length \((I,R)\)-modules to the category of finite-length \((g,K)\)-modules. If \( I \) is \( \theta \)-stable, this is exactly the functor of parabolic induction defined by Vogan in [16].

Now suppose \( O_q \subset N_q \) is a nilpotent \( G \)-orbit which is birationally induced from a nilpotent \( L \)-orbit \( O_l \subset N_l \). Let us say that a \((g,K)\)-module \textit{degenerate} if it has the form \( I_q^p(W) \) for a parabolic \( q' \subset g \), \( G \)-conjugate to \( q \) and a unipotent \((l',R')\)-module \( W \) (since \((l,R)\) is usually not a reductive pair, some care is required to make sense of this). Write \( \text{Deg}_q^p(O_l) \) for the set of (isomorphism classes of) degenerate representations.

In my thesis, I prove

\[\textbf{Theorem 4} \text{ (Mason-Brown). The sets } \text{Deg}_q^p(O_l) \text{ and } \text{Unip}(O_q) \text{ are related in the Grothendieck group of finite-length } (g,K) \text{-modules by an upper triangular change of basis matrix with } \pm 1 \text{ along the diagonal.}\]

Theorem 4 suggests a strategy for understanding unipotent representations in general: (1) understand the unipotent representations attached to non-induced orbits and (2) compute the change-of-basis matrices appearing in Theorem 4. Theorem 3 shows that this strategy is feasible (at least in some special cases).

Lusztig-Vogan Theory for Real Reductive Groups. In [5], Bezrukavnikov develops a theory of perverse coherent sheaves. Roughly speaking, he defines a \( t \)-structure on the derived category of constructible sheaves on a stratified space. His theory hands us a finite collection of canonical classes in the equivariant \( K \)-theory of \( N \). These classes play a central role in the Lusztig-Vogan bijection (conjecture: [11], [15] and proof: [5]). The analogue, for \( G \), of \( N \) is the closed subvariety \( N \cap p \). Adams and Vogan have conjectured ([11]) the existence of a similar correspondence for \( N \cap p \):

\[\textbf{Conjecture 2. The Grothendieck group } K \text{Coh}^K(N \cap p) \text{ admits two natural bases: one coming from the representation theory of } G \text{ (something like pairs } (H, \chi) \text{ consisting of a } \theta \text{-stable Cartan subgroup } H \text{ and a character } \chi \text{ of } H^p) \text{ and another coming from algebraic geometry (something like pairs } (\mathcal{O}, \mathcal{V}) \text{ consisting of a } K \text{-orbit } O \subset N \cap p \text{ and an irreducible } K \text{-equivariant vector bundle } \mathcal{V} \to \mathcal{O}). \text{ There is a natural bijection between these two bases which is implemented by a geometric construction in the vein of Bezrukavnikov’s perverse coherent extension ([5]).}\]

Such a correspondence would provide deep insight into the representation theory of \( G \). Unfortunately, in the real setting, the ideas of Bezrukavnikov do not straightforwardly apply (his construction requires a codimension condition on the smooth strata, which is rarely satisfied in the case of \( N \cap p \)). The theory of unipotent representations offers an alternative approach. If \( X \) is any representation, one can define an associated class \( \text{gr}(X) \) in the equivariant \( K \)-theory of \( N \cap p \). In this manner, the unipotent representations of \( G \) provide a finite collection of canonical classes in \( K \text{Coh}^K(N \cap p) \). This raises an important and interesting question

\[\textbf{Question 1. Can we find a geometric description of the classes } \text{gr}(X) \in K \text{Coh}^K(N \cap p) \text{ for representations } X \in \text{Unip}(O) ?\]

Conjecture 1 (and Theorem 2) suggest an answer to this question when \( O \) has small boundary. Theorem 3 (and its Corollary, Equation 2) provides an answer when \( O = O^p \).

If \( X \) is a finite-length \((g,K)\)-module, \( AV(X) \) contains a ‘codimension-1’ skeleton

\[
j : AV(X) \subset AV(X)
\]
obtained by removing the $K$-orbits on $\text{AV}(X)$ of codimension $\geq 2$. There are strong reasons to believe that if $X$ is unipotent, then $\text{gr}(X)$ is a $j_*$ extension of its restriction to $\text{AV}(X)'$ (Theorem 2 is one such reason). Hence, Question 1 leads to

**Question 2.** If $Z \subset N \cap p$ is an equidimensional subvariety containing $K$-orbits of codimension 0 and 1, can we describe the Grothendieck group of equivariant sheaves on $Z$?

In joint work (in progress) with MIT graduate student James Tao, we provide an answer to this question when $Z$ has sufficiently nice singularities. Our result is based on a microlocal description of equivariant, coherent sheaves on $Z$ in the vein of Gelfand, MacPherson, and Vilonen ([6]). This description suggests a natural answer to Question 1 in a wide range of cases.

**References**