Hint: Condition on $|T|$.

3. Show that if $P[|X| \le 1] = 1$, then $\text{Var}(X) \le 1$ with equality iff $X = \pm 1$ with probability $\frac{1}{2}$.

Hint: $\text{Var}(X) \le EX^2$.

4. *Comparison of Bounds:* Both the Hoeffding and Chebychev bounds are functions of n and ϵ through $\sqrt{n}\epsilon$.

(a) Show that the ratio of the Hoeffding function $h(\sqrt{n}\epsilon)$ to the Chebychev function $c(\sqrt{n}\epsilon)$ tends to 0 as $\sqrt{n}\epsilon \to \infty$ so that $h(\cdot)$ is arbitrarily better than $c(\cdot)$ in the tails.

(b) Show that the normal approximation $2\Phi\left(\frac{\sqrt{n}\epsilon}{\sigma}\right) - 1$ gives lower results than h in the tails if $P[|X| \le 1] = 1$ because, if $\sigma^2 \le 1$, $1 - \Phi(t) \sim \varphi(t)/t$ as $t \to \infty$.

Note: Hoeffding (1963) exhibits better bounds for known σ^2.

5. Suppose $\lambda : R \to R$ has $\lambda(0) = 0$, is bounded, and has a bounded second derivative λ''. Show that if $X_1, \dots, X_n$ are i.i.d., $EX_1 = \mu$ and $\text{Var}\, X_1 = \sigma^2 < \infty$, then

$$E\lambda(\bar{X} - \mu) = \lambda'(0)\frac{\sigma}{\sqrt{n}}\sqrt{\frac{2}{\pi}} + O\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

Hint: $\sqrt{n}E(\lambda(|\bar{X} - \mu|) - \lambda(0)) = E\lambda'(0)\sqrt{n}|\bar{X} - \mu| + E\left(\frac{\lambda''}{2}(\widetilde{X} - \mu)(\bar{X} - \mu)^2\right)$ where $|\widetilde{X} - \mu| \le |\bar{X} - \mu|$. The last term is $\le \sup_x |\lambda''(x)|\sigma^2/n$ and the first tends to $\lambda'(0)\sigma\int_{-\infty}^{\infty} |z|\varphi(z)dz$ by Remark B.7.1(2).

Problems for Section 5.2

1. Using the notation of Theorem 5.2.1, show that

$$\sup\{P_{\mathbf{p}}(|\widehat{p}_n - \mathbf{p}| \ge \delta) : \mathbf{p} \in \mathcal{S}\} \le k/4n\delta^2.$$

2. Let $X_1, \dots, X_n$ be i.i.d. $\mathcal{N}(\mu, \sigma^2)$. Show that for all $n \ge 1$, all $\epsilon > 0$

$$\sup_{\sigma} P_{(\mu,\sigma)}[|\bar{X} - \mu| \ge \epsilon] = 1.$$

Hint: Let $\sigma \to \infty$.

3. Establish (5.2.5).

Hint: $|\widehat{q}_n - q(\mathbf{p})| \ge \epsilon \Rightarrow |\widehat{p}_n - \mathbf{p}| \ge \omega^{-1}(\epsilon)$.

4. Let (U_i, V_i), $1 \le i \le n$, be i.i.d. $\sim P \in \mathcal{P}$.

(a) Let $\gamma(P) = P[U_1 > 0, V_1 > 0]$. Show that if $P = \mathcal{N}(0, 0, 1, 1, \rho)$, then

$$\rho = \sin 2\pi\left(\gamma(P) - \frac{1}{4}\right).$$

(b) Deduce that if P is the bivariate normal distribution, then

$$\widetilde{\rho} \equiv \sin\left\{2\pi\left(\frac{1}{n}\sum_{i=1}^{n} 1(X_i > \bar{X})1(Y_i > \bar{Y})\right)\right\}$$

is a consistent estimate of ρ.

(c) Suppose $\rho(P)$ is defined generally as $\text{Cov}_P(U,V)/\sqrt{\text{Var}_P U \, \text{Var}_P V}$ for $P \in \mathcal{P} = \{P : E_P U^2 + E_P V^2 < \infty, \ \text{Var}_P U \, \text{Var}_P V > 0\}$. Show that the sample correlation coefficient continues to be a consistent estimate of $\rho(P)$ but $\widetilde{\rho}$ is no longer consistent.

5. Suppose $X_1, \dots, X_n$ are i.i.d. $\mathcal{N}(\mu, \sigma_0^2)$ where σ_0 is known.

(a) Show that condition (5.2.8) fails even in this simplest case in which $\bar{X} \xrightarrow{P} \mu$ is clear.
Hint: $\sup_\mu \left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{(X_i-\mu)^2}{\sigma_0^2} - \left(1 + \frac{(\mu-\mu_0)^2}{\sigma_0^2}\right)\right)\right| = \infty$.

(b) Show that condition (5.2.14)(i),(ii) holds.
Hint: K can be taken as $[-A, A]$, where A is an arbitrary positive and finite constant.

6. Prove that (5.2.14)(i) and (ii) suffice for consistency.

7. (Wald) Suppose $\theta \to \rho(X,\theta)$ is continuous, $\theta \in R$ and

(i) For some $\epsilon(\theta_0) > 0$

$$E_{\theta_0} \sup\{|\rho(X,\theta') - \rho(X,\theta)| : |\theta - \theta'| \le \epsilon(\theta_0)\} < \infty.$$

(ii) $E_{\theta_0} \inf\{\rho(X,\theta) - \rho(X,\theta_0) : |\theta - \theta_0| \ge A\} > 0$ for some $A < \infty$.

Show that the maximum contrast estimate $\widehat{\theta}$ is consistent.
Hint: From continuity of ρ, (i), and the dominated convergence theorem,

$$\lim_{\delta \to 0} E_{\theta_0} \sup\{|\rho(X,\theta') - \rho(X,\theta)| : \theta' \in S(\theta,\delta)\} = 0$$

where $S(\theta,\delta)$ is the δ ball about θ. Therefore, by the basic property of maximum contrast estimates, for each $\theta \ne \theta_0$, and $\epsilon > 0$ there is $\delta(\theta) > 0$ such that

$$E_{\theta_0} \inf\{\rho(X,\theta') - \rho(X,\theta_0) : \theta' \in S(\theta,\delta(\theta))\} > \epsilon.$$

By compactness there is a finite number $\theta_1, \dots, \theta_r$ of sphere centers such that

$$K \cap \{\theta : |\theta - \theta_0| \ge \lambda\} \subset \bigcup_{j=1}^{r} S(\theta_j, \delta(\theta_j)).$$

Now

$$\inf\left\{\frac{1}{n}\sum_{i=1}^{n}\{\rho(X_i,\theta) - \rho(X_i,\theta_0)\} : \theta \in K \cap \{\theta : |\theta - \theta_0| \ge \lambda\right\}$$

$$\geq \min_{1\leq j\leq r}\left\{\frac{1}{n}\sum_{i=1}^{n}\inf\{\rho(X_i,\theta')-\rho(X_i,\theta_0)\}:\theta'\in S(\theta_j,\delta(\theta_j))\right\}.$$

For r fixed apply the law of large numbers.

8. The condition of Problem 7(ii) can also fail. Let X_i be i.i.d. $\mathcal{N}(\mu,\sigma^2)$. Compact sets K can be taken of the form $\{|\mu|\leq A, \epsilon\leq\sigma\leq 1/\epsilon, \epsilon>0\}$. Show that the log likelihood tends to ∞ as $\sigma\to 0$ and the condition fails.

9. Indicate how the conditions of Problem 7 have to be changed to ensure uniform consistency on K.

10. Extend the result of Problem 7 to the case $\theta\in R^p$, $p>1$.

Problems for Section 5.3

1. Establish (5.3.9) in the exponential model of Example 5.3.1.

2. Establish (5.3.3) for j odd as follows:

(i) Suppose $X_1',\dots,X_n'$ are i.i.d. with the same distribution as $X_1,\dots,X_n$ but independent of them, and let $\bar{X}'=n^{-1}\Sigma X_i'$. Then $E|\bar{X}-\mu|^j\leq E|\bar{X}-\bar{X}'|^j$.

(ii) If ϵ_i are i.i.d. and take the values ± 1 with probability $\frac{1}{2}$, and if $c_1,\dots,c_n$ are constants, then by Jensen's inequality, for some constants M_j,

$$E\left|\sum_{i=1}^{n}c_i\epsilon_i\right|^j\leq E^{\frac{j}{j+1}}\left(\sum_{i=1}^{n}c_i\epsilon_i\right)^{j+1}\leq M_j\left(\sum_{i=1}^{n}c_i^2\right)^{\frac{j}{2}}.$$

(iii) Condition on $|X_i-X_i'|$, $i=1,\dots,n$, in (i) and apply (ii) to get

(iv) $E\left|\sum_{i=1}^n(X_i-X_i')\right|^j\leq M_jE\left[\sum_{i=1}^n(X_i-X_i')^2\right]^{\frac{j}{2}}\leq M_jn^{\frac{j}{2}}E\left[\frac{1}{n}\sum_{i=1}^n(X_i-X_i')^2\right]^{\frac{j}{2}}\leq M_jn^{\frac{j}{2}}E\left(\frac{1}{n}\sum|X_i-X_i'|^j\right)\leq M_jn^{\frac{j}{2}}E|X_1-\mu|^j$.

3. Establish (5.3.11).
Hint: See part (a) of the proof of Lemma 5.3.1.

4. Establish Theorem 5.3.2. *Hint:* Taylor expand and note that if $i_1+\cdots+i_d=m$

$$\begin{aligned}E\left|\prod_{k=1}^{d}(\bar{Y}_k-\mu_k)^{i_k}\right| &\leq m^{m+1}\sum_{k=1}^{d}E|\bar{Y}_k-\mu_k|^m\\ &\leq C_mn^{-k/2}.\end{aligned}$$

Suppose $a_d \geq 0$, $1 \leq j \leq m$, $\sum_{j=1}^{d} i_j = m$ then

$$a_1^{i_1}, \ldots, a_d^{i_d} \leq [\max(a_1, \ldots, a_d)]^m \leq \left(\sum_{j=1}^{m} a_j\right)^m$$
$$\leq m^{m-1} \sum_{j=1}^{m} a_j^m.$$

5. Let $X_1, \ldots, X_n$ be i.i.d. R valued with $EX_1 = 0$. Show that

$$\sup\{|E(X_{i_1}, \ldots, X_{i_j})| : i_1, \ldots, i_j; j = 1, \ldots, n\} = E|X_1|.$$

6. Show that if $E|X_1|^j < \infty$, $j \geq 2$, then $E|X_1 - \mu|^j \leq 2^j E|X_1|^j$.
Hint: By the iterated expectation theorem

$$\begin{aligned} E|X_1 - \mu|^j &= E\{|X_1 - \mu|^j \mid |X_1| \geq |\mu|\} P(|X_1| \geq |\mu|) \\ &\quad + E\{|X_1 - \mu|^j \mid |X_1| < |\mu|\} P(|X_1| < |\mu|). \end{aligned}$$

7. Establish 5.3.28.

8. Let $X_1, \ldots, X_{n_1}$ be i.i.d. F and $Y_1, \ldots, Y_{n_2}$ be i.i.d. G, and suppose the X's and Y's are independent.

(a) Show that if F and G are $\mathcal{N}(\mu_1, \sigma_1^2)$ and $\mathcal{N}(\mu_2, \sigma_2^2)$, respectively, then the LR test of $H : \sigma_1^2 = \sigma_2^2$ versus $K : \sigma_1^2 \neq \sigma_2^2$ is based on the statistic s_1^2/s_2^2, where $s_1^2 = (n_1 - 1)^{-1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2$, $s_2^2 = (n_2 - 1)^{-1} \sum_{j=1}^{n_2} (Y_j - \bar{Y})^2$.

(b) Show that when F and G are normal as in part (a), then $(s_1^2/\sigma_1^2)/(s_2^2/\sigma_2^2)$ has an $\mathcal{F}_{k,m}$ distribution with $k = n_1 - 1$ and $m = n_2 - 1$.

(c) Now suppose that F and G are not necessarily normal but that

$$G \in \mathcal{G} = \left\{F\left(\frac{\cdot - a}{b}\right) : a \in R,\ b > 0\right\}$$

and that $0 < \mathrm{Var}(X_1^2) < \infty$. Show that if $m = \lambda k$ for some $\lambda > 0$ and

$$c_{k,m} = 1 + \sqrt{\frac{\kappa(k+m)}{km}}\, z_{1-\alpha},\ \kappa = \mathrm{Var}[(X_1 - \mu_1)/\sigma_1]^2,\ \mu_1 = E(X_1),\ \sigma_1^2 = \mathrm{Var}(X_1).$$

Then, under $H : \mathrm{Var}(X_1) = \mathrm{Var}(Y_1)$, $P(s_1^2/s_2^2 \leq c_{k,m}) \to 1 - \alpha$ as $k \to \infty$.

(d) Let $\widehat{c}_{k,m}$ be $c_{k,m}$ with κ replaced by its method of moments estimate. Show that under the assumptions of part (c), if $0 < EX_1^8 < \infty$, $P_H(s_1^2/s_2^2 \leq \widehat{c}_{k,m}) \to 1 - \alpha$ as $k \to \infty$.

(e) Next drop the assumption that $G \in \mathcal{G}$. Instead assume that $0 < \text{Var}(Y_1^2) < \infty$. Under the assumptions of part (c), use a normal approximation to find an approximate critical value $q_{k,m}$ (depending on $\kappa_1 = \text{Var}[(X_1 - \mu_1)/\sigma_1]^2$ and $\kappa_2 = \text{Var}[(X_2 - \mu_2)/\sigma_2]^2$ such that $P_H(s_1^2/s_2^2 \le q_{k,m}) \to 1 - \alpha$ as $k \to \infty$.

(f) Let $\widehat{q}_{k,m}$ be $q_{k,m}$ with κ_1 and κ_2 replaced by their method of moment estimates. Show that under the assumptions of part (e), if $0 < EX_1^8 < \infty$ and $0 < EY_1^8 < \infty$, then $P(s_1^2/s_2^2 \le \widehat{q}_{k,m}) \to 1 - \alpha$ as $k \to \infty$.

9. In Example 5.3.6, show that

(a) If $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\sqrt{n}((\widehat{C} - \rho), (\widehat{\sigma}_1^2 - 1), (\widehat{\sigma}_2^2 - 1))^T$ has the same asymptotic distribution as $n^{\frac{1}{2}}[n^{-1}\Sigma X_i Y_i - \rho, n^{-1}\Sigma X_i^2 - 1, n^{-1}\Sigma Y_i^2 - 1]^T$.

(b) If $(X, Y) \sim \mathcal{N}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then $\sqrt{n}(r^2 - \rho^2) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 4\rho^2(1 - \rho^2)^2)$ and, if $\rho \neq 0$, then $\sqrt{n}(r - \rho) \to \mathcal{N}(0, (1 - \rho^2)^2)$.

(c) Show that if $\rho = 0$, $\sqrt{n}(r - \rho) \to \mathcal{N}(0, 1)$.

Hint: Use the central limit theorem and Slutsky's theorem. Without loss of generality, $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$.

10. Show that $\frac{1}{2}\log\left(\frac{1+\rho}{1-\rho}\right)$ is the variance stabilizing transformation for the correlation coefficient in Example 5.3.6.

Hint: Write $\frac{1}{(1-\rho)^2} = \frac{1}{2}\left(\frac{1}{1-\rho} + \frac{1}{1+\rho}\right)$.

11. In survey sampling the *model-based* approach postulates that the population $\{x_1, \dots, x_N\}$ or $\{(u_1, x_1), \dots, (u_N, x_N)\}$ we are interested in is itself a sample from a superpopulation that is known up to parameters; that is, there exists $T_1, \dots, T_N$ i.i.d. P_θ, $\theta \in \Theta$ such that $T_i = t_i$ where $t_i \equiv (u_i, x_i)$, $i = 1, \dots, N$. In particular, suppose in the context of Example 3.4.1 that we use $T_{i_1}, \dots, T_{i_n}$, which we have sampled at random from $\{t_1, \dots, t_N\}$, to estimate $\bar{x} \equiv \frac{1}{N}\sum_{i=1}^N x_i$. Without loss of generality, suppose $i_j = j$, $1 \le j \le n$. Consider as estimates

(i) $\bar{X} = \frac{X_1 + \cdots + X_n}{n}$ when $T_i \equiv (U_i, X_i)$.

(ii) $\widehat{\bar{X}}_R = \widehat{b}_{\text{opt}}(\bar{U} - \bar{u})$ as in Example 3.4.1.

Show that, if $\frac{n}{N} \to \lambda$ as $N \to \infty$, $0 < \lambda < 1$, and if $EX_1^2 < \infty$ (in the supermodel), then

(a) $\sqrt{n}(\bar{X} - \bar{x}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau^2(1 - \lambda))$ where $\tau^2 = \text{Var}(X_1)$.

(b) Suppose P_θ is such that $X_1 = bU_i + \epsilon_i$, $i = 1, \dots, N$ where the ϵ_i are i.i.d., $E\epsilon_i = 0$, $\text{Var}(\epsilon_i) = \sigma^2 < \infty$ and $\text{Var}(U) > 0$. Show that $\sqrt{n}(\bar{X}_R - \bar{x}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, (1-\lambda)\sigma^2)$, $\sigma^2 < \tau^2$.

Hint: (a) $\bar{X} - \bar{x} = \left(1 - \frac{n}{N}\right)(\bar{X} - \bar{X}^c)$ where $\bar{X}^c = \frac{1}{N-n}\sum_{i=n+1}^N X_i$.

(b) Use the delta method for the multivariate case and note $(\widehat{b}_{\text{opt}} - b)(\bar{U} - \bar{u}) = o_p(n^{-\frac{1}{2}})$.

12. **(a)** Suppose that $E|Y_1|^3 < \infty$. Show that $|E(\bar{Y}_a - \mu_a)(\bar{Y}_b - \mu_b)(\bar{Y}_c - \mu_c)| \le Mn^{-2}$.

(b) Deduce formula (5.3.14).

Hint: If U is independent of (V, W), $EU = 0$, $E(WV) < \infty$, then $E(UVW) = 0$.

13. Let S_n have a χ^2_n distribution.

(a) Show that if n is large, $\sqrt{S_n} - \sqrt{n}$ has approximately a $\mathcal{N}(0, \frac{1}{2})$ distribution. This is known as *Fisher's approximation*.

(b) From (a) deduce the approximation $P[S_n \le x] \approx \Phi(\sqrt{2x} - \sqrt{2n})$.

(c) Compare the approximation of (b) with the central limit approximation $P[S_n \le x] = \Phi((x-n)/\sqrt{2n})$ and the exact values of $P[S_n \le x]$ from the χ^2 table for $x = x_{0.90}$, $x = x_{0.99}$, $n = 5, 10, 25$. Here x_q denotes the qth quantile of the χ^2_n distribution.

14. Suppose $X_1, \dots, X_n$ is a sample from a population with mean μ, variance σ^2, and third central moment μ_3. Justify formally

$$E[h(\bar{X}) - E(h(\bar{X}))]^3 = \frac{1}{n^2}[h'(\mu)]^3\mu_3 + \frac{3}{n^2}h''(\mu)[h'(\mu)]^2\sigma^4 + 0(n^{-3}).$$

Hint: Use (5.3.12).

15. It can be shown (under suitable conditions) that the normal approximation to the distribution of $h(\bar{X})$ improves as the coefficient of skewness γ_{1n} of $h(\bar{X})$ diminishes.

(a) Use this fact and Problem 5.3.14 to explain the numerical results of Problem 5.3.13(c).

(b) Let $S_n \sim \chi^2_n$. The following approximation to the distribution of S_n (due to Wilson and Hilferty, 1931) is found to be excellent

$$P[S_n \le x] \approx \Phi\left\{\left[\left(\frac{x}{n}\right)^{1/3} - 1 + \frac{2}{9n}\right]\sqrt{\frac{9n}{2}}\right\}.$$

Use (5.3.6) to explain why.

16. *Normalizing Transformation for the Poisson Distribution.* Suppose $X_1, \dots, X_n$ is a sample from a $\mathcal{P}(\lambda)$ distribution.

(a) Show that the only transformations h that make $E[h(\bar{X}) - E(h(\bar{X}))]^3 = 0$ to terms up to order $1/n^2$ for all $\lambda > 0$ are of the form $h(t) = ct^{2/3} + d$.

(b) Use (a) to justify the approximation

$$P\left[\bar{X} \le \frac{k}{n}\right] \approx \Phi\left\{\sqrt{n}\left[\left(\frac{k+\frac{1}{2}}{n}\right)^{2/3} - \lambda^{2/3}\right]\Big/\frac{2}{3}\lambda^{1/6}\right\}.$$

17. Suppose $X_1, \dots, X_n$ are independent, each with Hardy-Weinberg frequency function f given by

x	0	1	2
$f(x)$	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

where $0 < \theta < 1$.

(a) Find an approximation to $P[\bar{X} \leq t]$ in terms of θ and t.

(b) Find an approximation to $P[\sqrt{\bar{X}} \leq t]$ in terms of θ and t.

(c) What is the approximate distribution of $\sqrt{n}(\bar{X} - \mu) + \bar{X}^2$, where $\mu = E(X_1)$?

18. *Variance Stabilizing Transformation for the Binomial Distribution.* Let $X_1, \ldots, X_n$ be the indicators of n binomial trials with probability of success θ. Show that the only variance stabilizing transformation h such that $h(0) = 0$, $h(1) = 1$, and $h'(t) \geq 0$ for all t, is given by $h(t) = (2/\pi)\sin^{-1}(\sqrt{t})$.

19. Justify formally the following expressions for the moments of $h(\bar{X}, \bar{Y})$ where $(X_1, Y_1), \ldots, (X_n, Y_n)$ is a sample from a bivariate population with $E(X) = \mu_1$, $E(Y) = \mu_2$, $\text{Var}(X) = \sigma_1^2$, $\text{Var}(Y) = \sigma_2^2$, $\text{Cov}(X, Y) = \rho\sigma_1\sigma_2$.

(a)

$$E(h(\bar{X}, \bar{Y})) = h(\mu_1, \mu)2) + 0(n^{-1}).$$

(b)

$$\begin{aligned}\text{Var}(h(\bar{X}, \bar{Y})) \cong\ & \frac{1}{n}\{[h_1(\mu_1, \mu_2)]^2\sigma_1^2 \\ & +2h_1(\mu_1, \mu_2)h_2(\mu_1, \mu_2)\rho\sigma_1\sigma_2 + [h_2(\mu_1, \mu_2)]^2\sigma_2^2\} + 0(n^{-2})\end{aligned}$$

where

$$h_1(x, y) = \frac{\partial}{\partial x}h(x, y),\ h_2(x, y) = \frac{\partial}{\partial y}h(x, y).$$

Hint: $h(\bar{X}, \bar{Y}) - h(\mu_1, \mu_2) = h_1(\mu_1, \mu_2)(\bar{X} - \mu_1) + h_2(\mu_1, \mu_2)(\bar{Y} - \mu_2) + 0(n^{-1})$.

20. Let $B_{m,n}$ have a beta distribution with parameters m and n, which are integers. Show that if m and n are both tending to ∞ in such a way that $m/(m+n) \to \alpha$, $0 < \alpha < 1$, then

$$P\left[\sqrt{m+n}\frac{(B_{m,n} - m/(m+n))}{\sqrt{\alpha(1-\alpha)}} \leq x\right] \to \Phi(x).$$

Hint: Use $B_{m,n} = (m\bar{X}/n\bar{Y})[1 + (m\bar{X}/n\bar{Y})]^{-1}$ where $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ are independent standard exponentials.

21. Show directly using Problem B.2.5 that under the conditions of the previous problem, if $m/(m+n) - \alpha$ tends to zero at the rate $1/(m+n)^2$, then

$$E(B_{m,n}) = \frac{m}{m+n},\ \text{Var}\ B_{m,n} = \frac{\alpha(1-\alpha)}{m+n} + R_{m,n}$$

where $R_{m,n}$ tends to zero at the rate $1/(m+n)^2$.

22. Let $S_n \sim \chi_n^2$. Use Stirling's approximation and Problem B.2.4 to give a direct justification of

$$E(\sqrt{S_n}) = \sqrt{n} + R_n$$

where $R_n/\sqrt{n} \to 0$ as in $n \to \infty$. Recall *Stirling's approximation:*

$$\Gamma(p+1)/(\sqrt{2\pi}e^{-p}p^{p+\frac{1}{2}}) \to 1 \text{ as } p \to \infty.$$

(It may be shown but is not required that $|\sqrt{n}R_n|$ is bounded.)

23. Suppose that $X_1, \ldots, X_n$ is a sample from a population and that h is a real-valued function of $\bar{X}$ whose derivatives of order k are denoted by $h^{(k)}$, $k > 1$. Suppose $|h^{(4)}(x)| \le M$ for all x and some constant M and suppose that μ_4 is finite. Show that $Eh(\bar{X}) = h(\mu) + \frac{1}{2}h^{(2)}(\mu) + \frac{\sigma^2}{n} + R_n$ where $|R_n| \le h^{(3)}(\mu)|\mu_3|/6n^2 + M(\mu_4 + 3\sigma^2)/24n^2$.

Hint:

$$\left|h(x) - h(\mu) - h^{(1)}(\mu)(x-\mu) - \frac{h^{(2)}(\mu)}{2}(x-\mu)^2 - \frac{h^{(3)}(\mu)}{6}(x-\mu)^3\right| \le \frac{M}{24}(x-\mu)^4.$$

Therefore,

$$\begin{aligned}
&\left|Eh(\bar{X}) - h(\mu) - h^{(1)}(\mu)E(\bar{X}-\mu) - \frac{h^{(2)}(\mu)}{2}E(\bar{X}-\mu)^2\right| \\
&\le \frac{|h^{(3)}(\mu)|}{6}|E(\bar{X}-\mu)^3| + \frac{M}{24}E(\bar{X}-\mu)^4 \\
&\le \frac{|h^{(3)}(\mu)|}{6}\frac{|\mu_3|}{n^2} + \frac{M}{24}\frac{(\mu_4 + 3\sigma^4)}{n^2}.
\end{aligned}$$

24. Let $X_1, \ldots, X_n$ be a sample from a population with mean μ and variance $\sigma^2 < \infty$. Suppose h has a second derivative $h^{(2)}$ continuous at μ and that $h^{(1)}(\mu) = 0$.

(a) Show that $\sqrt{n}[h(\bar{X}) - h(\mu)] \to 0$ while $n[h(\bar{X} - h(\mu)]$ is asymptotically distributed as $\frac{1}{2}h^{(2)}(\mu)\sigma^2 V$ where $V \sim \chi_1^2$.

(b) Use part (a) to show that when $\mu = \frac{1}{2}$, $n[\bar{X}(1-\bar{X}) - \mu(1-\mu)] \xrightarrow{\mathcal{L}} -\sigma^2 V$ with $V \sim \chi_1^2$. Give an approximation to the distribution of $\bar{X}(1-\bar{X})$ in terms of the χ_1^2 distribution function when $\mu = \frac{1}{2}$.

25. Let $X_1, \ldots, X_n$ be a sample from a population with $\sigma^2 = \text{Var}(X) < \infty$, $\mu = E(X)$ and let $T = \bar{X}^2$ be an estimate of μ^2.

(a) When $\mu \ne 0$, find the asymptotic distribution of $\sqrt{n}(T - \mu^2)$ using the delta method.

(b) When $\mu = 0$, find the asymptotic distriution of nT using $P(nT \le t) = P(-\sqrt{t} \le \sqrt{n}\bar{X} \le \sqrt{t})$. Compare your answer to the answer in part (a).

(c) Find the limiting laws of $\sqrt{n}(\bar{X} - \mu)^2$ and $n(\bar{X} - \mu)^2$.