8. **(a)** Show that the power $P_F[D_n \geq k_\alpha]$ of the Kolmogorov test is bounded below by

$$\sup_x P_F[|\widehat{F}(x) - F_0(x)| \geq k_\alpha].$$

Hint: $D_n \geq |\widehat{F}(x) - F_0(x)|$ for each x.

(b) Suppose F_0 is $\mathcal{N}(0,1)$ and $F(x) = (1+\exp(-x/\tau))^{-1}$ where $\tau = \sqrt{3}/\pi$ is chosen so that $\int_{-\infty}^{\infty} x^2 dF(x) = 1$. (This is the logistic distribution with mean zero and variance 1.) Evaluate the bound $P_F(|\widehat{F}(x) - F_0(x)| \geq k_\alpha)$ for $\alpha = 0.10$, $n = 80$ and $x = 0.5$, 1, and 1.5 using the normal approximation to the binomial distribution of $n\widehat{F}(x)$ and the approximate critical value in Example 4.1.5.

(c) Show that if F and F_0 are continuous and $F \neq F_0$, then the power of the Kolmogorov test tends to 1 as $n \to \infty$.

9. Let $X_1, \ldots, X_n$ be i.i.d. with distribution function F and consider $H : F = F_0$. Suppose that the distribution $\mathcal{L}_0$ of the statistic $T = T(\mathbf{X})$ is continuous under H and that H is rejected for large values of T. Let $T^{(1)}, \ldots, T^{(B)}$ be B independent Monte Carlo simulated values of T. (In practice these can be obtained by drawing B independent samples $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(B)}$ from F_0 on the computer and computing $T^{(j)} = T(\mathbf{X}^{(j)})$, $j = 1, \ldots, B$. Here, to get X with distribution F_0, generate a $\mathcal{U}(0,1)$ variable on the computer and set $X = F_0^{-1}(U)$ as in Problem B.2.12(b).) Next let $T^{(1)}, \ldots, T^{(B+1)}$ denote $T, T^{(1)}, \ldots, T^{(B)}$ ordered. Show that the test rejects H iff $T \geq T^{(B+1-m)}$ has level $\alpha = m/(B+1)$.

Hint: If H is true $T(\mathbf{X}), T(\mathbf{X}^{(1)}), \ldots, T(\mathbf{X}^{(B)})$ is a sample of size $B+1$ from $\mathcal{L}_0$. Use the fact that $T(\mathbf{X})$ is equally likely to be any particular order statistic.

10. **(a)** Show that the statistic T_n of Example 4.1.6 is invariant under location and scale. That is, if $X_i' = (X_i - a)/b$, $b > 0$, then $T_n(\mathbf{X}') = T_n(\mathbf{X})$.

(b) Use part (a) to conclude that $\mathcal{L}_{\mathcal{N}(\mu,\sigma^2)}(T_n) = \mathcal{L}_{\mathcal{N}(0,1)}(T_n)$.

11. In Example 4.1.5, let $\psi(u)$ be a function from $(0,1)$ to $(0,\infty)$, and let $\alpha > 0$. Define the statistics

$$\begin{aligned}
S_{\psi,\alpha} &= \sup_x \psi(F_0(x))|\widehat{F}(x) - F_0(x)|^\alpha \\
T_{\psi,\alpha} &= \sup_x \psi(\widehat{F}(x))|\widehat{F}(x) - F_0(x)|^\alpha \\
U_{\psi,\alpha} &= \int \psi(F_0(x))|\widehat{F}(x) - F_0(x)|^\alpha dF_0(x) \\
V_{\psi,\alpha} &= \int \psi(\widehat{F}(x))|\widehat{F}(x) - F_0(x)|^\alpha d\widehat{F}(x).
\end{aligned}$$

(a) For each of these statistics show that the distribution under H does not depend on F_0.

(b) When $\psi(u) = 1$ and $\alpha = 2$, $V_{\psi,\alpha}$ is called the Cramer–von Mises statistic. Express the Cramer–von Mises statistic as a sum.

(c) Are any of the four statistics in (a) invariant under location and scale. (See Problem 4.1.10.)

12. *Expected p-values.* Consider a test with critical region of the form $\{T \geq c\}$ for testing $H : \theta = \theta_0$ versus $K : \theta > \theta_0$. Without loss of generality, take $\theta_0 = 0$. Suppose that T has a continuous distribution F_θ, then the p-value is

$$U = 1 - F_0(T).$$

(a) Show that if the test has level α, the power is

$$\beta(\theta) = P(U \leq \alpha) = 1 - F_\theta(F_0^{-1}(1-\alpha))$$

where $F_0^{-1}(u) = \inf\{t : F_0(t) \geq u\}$.

(b) Define the expected p-value as $EPV(\theta) = E_\theta U$. Let T_0 denote a random variable with distribution F_0, which is independent of T. Show that $EPV(\theta) = P(T_0 \geq T)$.

Hint: $P(T_0 \geq T) = \int P(T_0 \geq t \mid T = t) f_\theta(t) dt$ where $f_\theta(t)$ is the density of $F_\theta(t)$.

(c) Suppose that for each $\alpha \in (0,1)$, the UMP test is of the form $1\{T \geq c\}$. Show that the $EPV(\theta)$ for $1\{T \geq c\}$ is uniformly minimal in $\theta > 0$ when compared to the $EPV(\theta)$ for any other test.

Hint: $P(T \leq t_0 \mid T_0 = t_0)$ is 1 minus the power of a test with critical value t_0.

(d) Consider the problem of testing $H : \mu = \mu_0$ versus $K : \mu > \mu_0$ on the basis of the $\mathcal{N}(\mu, \sigma^2)$ sample $X_1, \dots, X_n$, where σ is known. Let $T = \bar{X} - \mu_0$ and $\theta = \mu - \mu_0$. Show that $EPV(\theta) = \Phi(-\sqrt{n}\theta/\sqrt{2}\sigma)$, where Φ denotes the standard normal distribution. (For a recent review of expected p values see Sackrowitz and Samuel–Cahn, 1999.)

Problems for Section 4.2

1. Consider Examples 3.3.2 and 4.2.1. You want to buy one of two systems. One has signal-to-noise ratio $v/\sigma_0 = 2$, the other has $v/\sigma_0 = 1$. The first system costs $\$10^6$, the other $\$10^5$. One second of transmission on either system costs $\$10^3$ each. Whichever system you buy during the year, you intend to test the satellite 100 times. If each time you test, you want the number of seconds of response sufficient to ensure that both probabilities of error are ≤ 0.05, which system is cheaper on the basis of a year's operation?

2. Consider a population with three kinds of individuals labeled 1, 2, and 3 occuring in the Hardy–Weinberg proportions $f(1,\theta) = \theta^2$, $f(2,\theta) = 2\theta(1-\theta)$, $f(3,\theta) = (1-\theta)^2$. For a sample $X_1, \dots, X_n$ from this population, let N_1, N_2, and N_3 denote the number of X_j equal to 1, 2, and 3, respectively. Let $0 < \theta_0 < \theta_1 < 1$.

(a) Show that $L(\mathbf{x}, \theta_0, \theta_1)$ is an increasing function of $2N_1 + N_2$.

(b) Show that if $c > 0$ and $\alpha \in (0,1)$ satisfy $P_{\theta_0}[2N_1 + N_2 \geq c] = \alpha$, then the test that rejects H if, and only if, $2N_1 + N_2 \geq c$ is MP for testing $H : \theta = \theta_0$ versus $K : \theta = \theta_1$.

3. A gambler observing a game in which a single die is tossed repeatedly gets the impression that 6 comes up about 18% of the time, 5 about 14% of the time, whereas the other four numbers are equally likely to occur (i.e., with probability .17). Upon being asked to play, the gambler asks that he first be allowed to test his hypothesis by tossing the die n times.

(a) What test statistic should he use if the only alternative he considers is that the die is fair?

(b) Show that if $n = 2$ the most powerful level .0196 test rejects if, and only if, two 5's are obtained.

(c) Using the fact that if $(N_1, \dots, N_k) \sim \mathcal{M}(n, \theta_1, \dots, \theta_k)$, then $a_1N_1 + \cdots + a_kN_k$ has approximately a $\mathcal{N}(n\mu, n\sigma^2)$ distribution, where $\mu = \sum_{i=1}^k a_i\theta_i$ and $\sigma^2 = \sum_{i=1}^k \theta_i(a_i - \mu)^2$, find an approximation to the critical value of the MP level α test for this problem.

4. A formulation of goodness of tests specifies that a test is best if the maximum probability of error (of either type) is as small as possible.

(a) Show that if in testing $H : \theta = \theta_0$ versus $K : \theta = \theta_1$ there exists a critical value c such that

$$P_{\theta_0}[L(\mathbf{X}, \theta_0, \theta_1) \geq c] = 1 - P_{\theta_1}[L(\mathbf{X}, \theta_0, \theta_1) \geq c]$$

then the likelihood ratio test with critical value c is best in this sense.

(b) Find the test that is best in this sense for Example 4.2.1.

5. A newly discovered skull has cranial measurements (X, Y) known to be distributed either (as in population 0) according to $\mathcal{N}(0, 0, 1, 1, 0.6)$ or (as in population 1) according to $\mathcal{N}(1, 1, 1, 1, 0.6)$ where all parameters are known. Find a statistic $T(X, Y)$ and a critical value c such that if we use the classification rule, (X, Y) belongs to population 1 if $T \geq c$, and to population 0 if $T < c$, then the maximum of the two *probabilities of misclassification* $P_0[T \geq c]$, $P_1[T < c]$ is as small as possible.

Hint: Use Problem 4.2.4 and recall (Proposition B.4.2) that linear combinations of bivariate normal random variables are normally distributed.

6. Show that if randomization is permitted, MP-sized α likelihood ratio tests with $0 < \alpha < 1$ have power nondecreasing in the sample size.

7. Prove Corollary 4.2.1.

Hint: The MP test has power at least that of the test with test function $\delta(x) = \alpha$.

8. In Examle 4.2.2, derive the UMP test defined by (4.2.7).

9. In Example 4.2.2, if $\mathbf{\Delta}_0 = (1, 0, \dots, 0)^T$ and $\Sigma_0 \neq I$, find the MP test for testing $H : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ versus $K : \boldsymbol{\theta} = \boldsymbol{\theta}_1$.

10. For $0 < \alpha < 1$, prove Theorem 4.2.1(a) using the connection between likelihood ratio tests and Bayes tests given in Remark 4.2.1.

Problems for Section 4.3

1. Let X_i be the number of arrivals at a service counter on the ith of a sequence of n days. A possible model for these data is to assume that customers arrive according to a homogeneous Poisson process and, hence, that the X_i are a sample from a Poisson distribution with parameter θ, the expected number of arrivals per day. Suppose that if $\theta \leq \theta_0$ it is not worth keeping the counter open.

(a) Exhibit the optimal (UMP) test statistic for $H : \theta \leq \theta_0$ versus $K : \theta > \theta_0$.

(b) For what levels can you exhibit a UMP test?

(c) What distribution tables would you need to calculate the power function of the UMP test?

2. Consider the foregoing situation of Problem 4.3.1. You want to ensure that if the arrival rate is ≤ 10, the probability of your deciding to stay open is ≤ 0.01, but if the arrival rate is ≥ 15, the probability of your deciding to close is also ≤ 0.01. How many days must you observe to ensure that the UMP test of Problem 4.3.1 achieves this? (Use the normal approximation.)

3. In Example 4.3.4, show that the power of the UMP test can be written as

$$\beta(\sigma) = G_n(\sigma_0^2 x_n(\alpha)/\sigma^2)$$

where G_n denotes the χ^2_{2n} distribution function.

4. Let $X_1, \dots, X_n$ be the times in months until failure of n similar pieces of equipment. If the equipment is subject to wear, a model often used (see Barlow and Proschan, 1965) is the one where $X_1, \dots, X_n$ is a sample from a Weibull distribution with density $f(x, \lambda) = \lambda c x^{c-1} e^{-\lambda x^c}$, $x > 0$. Here c is a known positive constant and $\lambda > 0$ is the parameter of interest.

(a) Show that $\sum_{i=1}^n X_i^c$ is an optimal test statistic for testing $H : 1/\lambda \leq 1/\lambda_0$ versus $K : 1/\lambda > 1/\lambda_0$.

(b) Show that the critical value for the size α test with critical region $[\sum_{i=1}^n X_i^c \geq k]$ is $k = x_{2n}(1-\alpha)/2\lambda_0$ where $x_{2n}(1-\alpha)$ is the $(1-\alpha)$th quantile of the χ^2_{2n} distribution and that the power function of the UMP level α test is given by

$$1 - G_{2n}(\lambda x_{2n}(1-\alpha)/\lambda_0)$$

where G_{2n} denotes the χ^2_{2n} distribution function.
Hint: Show that $X_i^c \sim \mathcal{E}(\lambda)$.

(c) Suppose $1/\lambda_0 = 12$. Find the sample size needed for a level 0.01 test to have power at least 0.95 at the alternative value $1/\lambda_1 = 15$. Use the normal approximation to the critical value and the probability of rejection.

5. Show that if $X_1, \dots, X_n$ is a sample from a *truncated binomial* distribution with

$$p(x, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} / [1 - (1-\theta)^n],\ x = 1, \dots, n,$$

then $\sum_{i=1}^{n} X_i$ is an optimal test statistic for testing $H : \theta = \theta_0$ versus $K : \theta > \theta_0$.

6. Let $X_1, \dots, X_n$ denote the incomes of n persons chosen at random from a certain population. Suppose that each X_i has the Pareto density

$$f(x, \theta) = c^{\theta}\theta x^{-(1+\theta)}, \ x > c$$

where $\theta > 1$ and $c > 0$.

(a) Express mean income μ in terms of θ.

(b) Find the optimal test statistic for testing $H : \mu = \mu_0$ versus $K : \mu > \mu_0$.

(c) Use the central limit theorem to find a normal approximation to the critical value of test in part (b).

Hint: Use the results of Theorem 1.6.2 to find the mean and variance of the optimal test statistic.

7. In the goodness-of-fit Example 4.1.5, suppose that $F_0(x)$ has a nonzero density on some interval (a, b), $-\infty \leq a < b \leq \infty$, and consider the alternative with distribution function $F(x, \theta) = F_0^{\theta}(x)$, $0 < \theta < 1$. Show that the UMP test for testing $H : \theta \geq 1$ versus $K : \theta < 1$ rejects H if $-2\Sigma \log F_0(X_i) \geq x_{1-\alpha}$, where $x_{1-\alpha}$ is the $(1-\alpha)$th quantile of the χ^2_{2n} distribution. (See Problem 4.1.6.) It follows that Fisher's method for combining p-values (see 4.1.6) is UMP for testing that the p-values are uniformly distributed against $F(u) = u^{\theta}, 0 < \theta < 1$.

8. Let the distribution of survival times of patients receiving a standard treatment be the known distribution F_0, and let $Y_1, \dots, Y_n$ be the i.i.d. survival times of a sample of patients receiving an experimental treatment.

(a) *Lehmann Alternative.* In Problem 1.1.12, we derived the model

$$G(y, \Delta) = 1 - [1 - F_0(y)]^{\Delta}, \ y > 0, \ \Delta > 0.$$

To test whether the new treatment is beneficial we test $H : \Delta \leq 1$ versus $K : \Delta > 1$. Assume that F_0 has a density f_0. Find the UMP test. Show how to find critical values.

(b) *Nabeya–Miura Alternative.* For the purpose of modeling, imagine a sequence $X_1, X_2, \dots$ of i.i.d. survival times with distribution F_0. Let N be a zero-truncated Poisson, $\mathcal{P}(\lambda)$, random variable, which is independent of $X_1, X_2, \dots$.

(i) Show that if we model the distribution of Y as $\mathcal{L}(\max\{X_1, \dots, X_N\})$, then

$$P(Y \leq y) = \frac{e^{\lambda F_0(y)} - 1}{e^{\lambda} - 1}, \ y > 0, \ \lambda \geq 0.$$

(ii) Show that if we model the distribution of Y as $\mathcal{L}(\min\{X_1, \dots, X_N\})$, then

$$P(Y \leq y) = \frac{e^{-\lambda F_0(y)} - 1}{e^{-\lambda} - 1}, \ y > 0, \ \lambda \geq 0.$$

(iii) Consider the model

$$G(y,\theta) = \frac{e^{\theta F_0(y)} - 1}{e^\theta - 1}, \ \theta \neq 0$$
$$= F_0(y), \ \theta = 0.$$

To see whether the new treatment is beneficial, we test $H : \theta \leq 0$ versus $K : \theta > 0$. Assume that F_0 has a density $f_0(y)$. Show that the UMP test is based on the statistic $\sum_{i=1}^n F_0(Y_i)$.

9. Let $X_1, \dots, X_n$ be i.i.d. with distribution function $F(x)$. We want to test whether F is exponential, $F(x) = 1 - \exp(-x)$, $x > 0$, or Weibull, $F(x) = 1 - \exp(-x^\theta)$, $x > 0$, $\theta > 0$. Find the MP test for testing $H : \theta = 1$ versus $K : \theta = \theta_1 > 1$. Show that the test is not UMP.

10. Show that under the assumptions of Theorem 4.3.2 the class of all Bayes tests is complete.

Hint: Consider the class of all Bayes tests of $H : \theta = \theta_0$ versus $K : \theta = \theta_1$ where $\pi\{\theta_0\} = 1 - \pi\{\theta_1\}$ varies between 0 and 1.

11. Show that under the assumptions of Theorem 4.3.1 and 0-1 loss, every Bayes test for $H : \theta \leq \theta_0$ versus $K : \theta > \theta_1$ is of the form δ_t for some t.

Hint: A Bayes test rejects (accepts) H if

$$\int_{\theta_1}^{\infty} p(x,\theta)d\pi(\theta) / \int_{-\infty}^{\theta_0} p(x,\theta)d\pi(\theta) \ \overset{>}{(<)} \ 1.$$

The left-hand side equals

$$\frac{\int_{\theta_1}^{\infty} L(x,\theta,\theta_0)d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(x,\theta,\theta_0)d\pi(\theta)}.$$

The numerator is an increasing function of $T(x)$, the denominator decreasing.

12. Show that under the assumptions of Theorem 4.3.2, $1 - \delta_t$ is UMP for testing $H : \theta \geq \theta_0$ versus $K : \theta < \theta_0$.

Problems for Section 4.4

1. Let $X_1, \dots, X_n$ be a sample from a normal population with unknown mean μ and unknown variance σ^2. Using a pivot based on $\Sigma_{i=1}^n (X_i - \bar{X})^2$,

(a) Show how to construct level $(1 - \alpha)$ confidence intervals of fixed finite length for $\log \sigma^2$.

(b) Suppose that $\Sigma_{i=1}^n (X_i - \bar{X})^2 = 16.52$, $n = 2$, $\alpha = 0.01$. What would you announce as your level $(1 - \alpha)$ UCB for σ^2?

2. Let $X_i = (\theta/2)t_i^2 + \epsilon_i$, $i = 1, \dots, n$, where the ϵ_i are independent normal random variables with mean 0 and known variance σ^2 (cf. Problem 2.2.1).

(a) Using a pivot based on the MLE $(2\Sigma_{i=1}^n t_i^2 X_i)/\Sigma_{i=1}^n t_i^4$ of θ, find a fixed length level $(1-\alpha)$ confidence interval for θ.

(b) If $0 \le t_i \le 1$, $i = 1, \dots, n$, but we may otherwise choose the t_i freely, what values should we use for the t_i so as to make our interval as short as possible for given α?

3. Let $X_1, \dots, X_n$ be as in Problem 4.4.1. Suppose that an experimenter thinking he knows the value of σ^2 uses a lower confidence bound for μ of the form $\underline{\mu}(\mathbf{X}) = \bar{X} - c$, where c is chosen so that the confidence level under the assumed value of σ^2 is $1-\alpha$. What is the actual confidence coefficient of $\underline{\mu}$, if σ^2 can take on all positive values?

4. Suppose that in Example 4.4.3 we know that $\theta \le 0.1$.

(a) Justify the interval $[\underline{\theta}, \min(\bar{\theta}, 0.1)]$ if $\underline{\theta} < 0.1$, $[0.1, 0.1]$ if $\underline{\theta} \ge 0.1$, where $\underline{\theta}, \bar{\theta}$ are given by (4.4.3).

(b) Calculate the smallest n needed to bound the length of the 95% interval of part (a) by 0.02. Compare your result to the n needed for (4.4.3).

5. Show that if $\underline{q}(\mathbf{X})$ is a level $(1-\alpha_1)$ LCB and $\bar{q}(\mathbf{X})$ is a level $(1-\alpha_2)$ UCB for $q(\theta)$, then $[\underline{q}(\mathbf{X}), \bar{q}(\mathbf{X})]$ is a level $(1-(\alpha_1+\alpha_2))$ confidence interval for $q(\theta)$. (Define the interval arbitrarily if $\underline{q} > \bar{q}$.)

Hint: Use (A.2.7).

6. Show that if $X_1, \dots, X_n$ are i.i.d. $\mathcal{N}(\mu, \sigma^2)$ and $\alpha_1 + \alpha_2 \le \alpha$, then the shortest level $(1-\alpha)$ interval of the form

$$\left[\bar{X} - z(1-\alpha_1)\frac{\sigma}{\sqrt{n}},\ \bar{X} + z(1-\alpha_2)\frac{\sigma}{\sqrt{n}}\right]$$

is obtained by taking $\alpha_1 = \alpha_2 = \alpha/2$ (assume σ^2 known).

Hint: Reduce to $\alpha_1 + \alpha_2 = \alpha$ by showing that if $\alpha_1 + \alpha_2 < \alpha$, there is a shorter interval with $\alpha_1 + \alpha_2 = \alpha$. Use calculus.

7. Suppose we want to select a sample size N such that the interval (4.4.1) based on $n = N$ observations has length at most l for some preassigned length $l = 2d$. Stein's (1945) two-stage procedure is the following. Begin by taking a fixed number $n_0 \ge 2$ of observations and calculate $\bar{X}_0 = (1/n_0)\Sigma_{i=1}^{n_0} X_i$ and

$$s_0^2 = (n_0 - 1)^{-1}\Sigma_{i=1}^{n_0}(X_i - \bar{X}_0)^2.$$

Then take $N - n_0$ further observations, with N being the smallest integer greater than n_0 and greater than or equal to

$$\left[s_0 t_{n_0-1}\left(1 - \tfrac{1}{2}\alpha\right)/d\right]^2.$$

Show that, although N is random, $\sqrt{N}(\bar{X} - \mu)/s_0$, with $\bar{X} = \Sigma_{i=1}^N X_i/N$, has a $\mathcal{T}_{n_0-1}$ distribution. It follows that

$$\left[\bar{X} - s_0 t_{n_0-1}\left(1 - \tfrac{1}{2}\alpha\right)/\sqrt{N},\ \bar{X} + s_0 t_{n_0-1}\left(1 - \tfrac{1}{2}\alpha\right)/\sqrt{N}\right]$$

is a confidence interval with confidence coefficient $(1-\alpha)$ for μ of length at most $2d$. (The sticky point of this approach is that we have no control over N, and, if σ is large, we may very likely be forced to take a prohibitively large number of observations. The reader interested in pursuing the study of sequential procedures such as this one is referred to the book of Wetherill and Glazebrook, 1986, and the fundamental monograph of Wald, 1947.)

Hint: Note that $\bar{X} = (n_0/N)\bar{X}_{n_0} + (1/N)\Sigma_{i=n_0+1}^{N} X_i$. By Theorem B.3.3, s_{n_0} is independent of $\bar{X}_{n_0}$. Because N depends only on s_{n_0}, given $N=k$, $\bar{X}$ has a $\mathcal{N}(\mu, \sigma^2/k)$ distribution. Hence, $\sqrt{N}(\bar{X}-\mu)$ has a $\mathcal{N}(0,\sigma^2)$ distribution and is independent of s_{n_0}.

8. **(a)** Show that in Problem 4.4.6, in order to have a level $(1-\alpha)$ confidence interval of length at most $2d$ when σ^2 is known, it is necessary to take at least $z^2\left(1-\frac{1}{2}\alpha\right)\sigma^2/d^2$ observations.

Hint: Set up an inequality for the length and solve for n.

(b) What would be the minimum sample size in part (a) if $\alpha = 0.001$, $\sigma^2 = 5$, $d = 0.05$?

(c) Suppose that σ^2 is not known exactly, but we are sure that $\sigma^2 \le \sigma_1^2$. Show that $n \ge z^2\left(1-\frac{1}{2}\alpha\right)\sigma_1^2/d^2$ observations are necessary to achieve the aim of part (a).

9. Let $S \sim \mathcal{B}(n,\theta)$ and $\bar{X} = S/n$.

(a) Use (A.14.18) to show that $\sin^{-1}(\sqrt{\bar{X}}) \pm z\left(1-\frac{1}{2}\alpha\right)/2\sqrt{n}$ is an approximate level $(1-\alpha)$ confidence interval for $\sin^{-1}(\sqrt{\theta})$.

(b) If $n=100$ and $\bar{X}=0.1$, use the result in part (a) to compute an approximate level 0.95 confidence interval for θ.

10. Let $X_1,\dots,X_{n_1}$ and $Y_1,\dots,Y_{n_2}$ be two independent samples from $\mathcal{N}(\mu,\sigma^2)$ and $\mathcal{N}(\eta,\tau^2)$ populations, respectively.

(a) If all parameters are unknown, find ML estimates of μ, ν, σ^2, τ^2. Show that these two quadruples are each sufficient.

(b) Exhibit a level $(1-\alpha)$ confidence interval for τ^2/σ^2 using a pivot based on the statistics of part (a). Indicate what tables you would need to calculate the interval.

(c) If σ^2, τ^2 are known, exhibit a fixed length level $(1-\alpha)$ confidence interval for $(\eta-\mu)$.

Such two sample problems arise in comparing the precision of two instruments and in determining the effect of a treatment.

11. Show that the endpoints of the approximate level $(1-\alpha)$ interval defined by (4.4.3) are indeed approximate level $\left(1-\frac{1}{2}\alpha\right)$ upper and lower bounds.

Hint: $[\underline{\theta}(\mathbf{X}) \le \theta] = \left[\sqrt{n}(\bar{X}-\theta)/[\theta(1-\theta)]^{\frac{1}{2}} \le z\left(1-\frac{1}{2}\alpha\right)\right]$.

12. Let $S \sim \mathcal{B}(n,\theta)$. Suppose that it is known that $\theta \le \frac{1}{4}$.

(a) Show that $\bar{X} \pm \sqrt{3}z\left(1-\frac{1}{2}\alpha\right)/4\sqrt{n}$ is an approximate level $(1-\alpha)$ confidence interval for θ.