18.466 Midterm 2

1. A company has manufactured certain objects and printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects that have been manufactured. A sample of three of these objects is drawn at random with replacement and the serial numbers X_1, X_2, X_3 for the sampled units are recorded.

(a) What is the method of moments estimator of N using the first moment?

(b) What is the maximum likelihood estimator of N?

(c) Compare the risk functions for squared error loss of the estimators you found in (b) and (c). Does one estimator dominate the other?

(d) Which of the two estimators, the method of moments or the maximum likelihood estimator, can be improved for squared error loss by using the Rao-Blackwell Theorem? Use the Rao-Blackwell Theorem to find an estimator that improves on the estimator that can be improved.

2. A loss function investigated by Zellner (1986, Journal of the American Statistical Association) is the LINEX (LINear-Exponential) loss, a loss function that can handle asymmetries in a smooth way. The LINEX loss is given by

$$l(\theta, a) = e^{c(a-\theta)} - c(a-\theta) - 1$$

where c is a positive constant. As the constant c varies, the loss function varies from very asymmetric to almost symmetric.

(a) For $c = 0.2, 0.5, 1$, plot $l(\theta, a)$ as a function of $a - \theta$.

(b) If $\boldsymbol{X} \sim p(\boldsymbol{x} \mid \theta)$, show that the Bayes estimator of θ using a prior π and LINEX loss is given by $\delta^{\pi}(\boldsymbol{X}) = -\frac{1}{c} \log E[e^{-c\theta} \mid \boldsymbol{X}]$ where $E[e^{-c\theta} \mid \boldsymbol{X}]$ is the expectation of $e^{-c\theta}$ under the posterior distribution of θ given the observed data $\boldsymbol{X}$.

(c) Let $X_1,\ldots,X_n$ be iid $N(\mu,\sigma^2)$, where σ^2 is known, and suppose that our prior distribution for μ is $N(\eta,\tau^2)$. Find the Bayes estimator of μ under LINEX loss.

3. Let X be a single sample from the geometric (θ) distribution $P(X=k\mid\theta)=(1-\theta)^k\theta$, $k=0,1,2,\ldots$ Show that $X+1$ is the uniformly minimum variance unbiased (UMVU) estimator of the parameter $1/\theta$.

4. Let X and Y be independent Binomial random variables with n_X and n_Y trials respectively and probabilities of success p_X and p_Y respectively, where n_X and n_Y are known and p_X and p_Y are unknown.

(a) Suppose it is known that $0 \le p_X \le p_Y \le 1$. Find the maximum likelihood estimates of p_X and p_Y.

(b) Find a most powerful level 0.05 test of $H_0: p_X = p_Y = 0.5$ vs. $H_1: p_X = 0.6,\ p_Y = 0.75$ where you can use the Central Limit Theorem to approximate the critical value for the test. Would this test accept or reject H_0 for $X=15, n_X=30, Y=21, n_Y=30$?

(c) For the parameter space $0 \le p_X \le p_Y \le 1$, does there exist a uniformly most powerful test of $H_0: p_X = p_Y = 0.5$ vs. $H_1: p_X < p_Y,\ p_X \ge 0.5$? Prove or disprove.

5. Let $\boldsymbol{X}=(X_1,\ldots,X_n)$ be an iid sample from a Uniform $(\theta,\theta+1)$ distribution.

(a) For testing $H_0: \theta \le \theta_0$ against $H_1: \theta > \theta_0$ at level α, show that there exists a uniformly most powerful (UMP) test which rejects when $\min(X_1, \ldots, X_n) > \theta_0 + C(\alpha)$ or $\max(X_1, \ldots, X_n) > \theta_0 + 1$ for suitable $C(\alpha)$ [find the suitable $C(\alpha)$ and show that the test is UMP].

(b) Show that the family of distributions $\boldsymbol{X} = (X_1, \ldots, X_n) \sim$ iid Uniform $(\theta, \theta+1)$, $-\infty < \theta < \infty$ does not have monotone likelihood ratio in any statistic $T(\boldsymbol{X})$. Hint: Show that if a family of distributions has monotone likelihood ratio, then the set of UMP level α tests of $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ generated as α varies from 0 to 1 must be the same for all θ_0.

Note: Theorem 4.3.1 shows that a family of distributions having monotone likelihood ratio is a sufficient condition for UMP tests to exist for one-sided testing problems. This problem shows that it is not a necessary condition.