1. For any density functions $p(x)$ and $q(x)$, define the total variation affinity

$$\eta(p,q) = \int p(x) \wedge q(x) dx.$$

   And the Hellinger affinity $\rho(p,q) = \int \sqrt{p(x)q(x)} dx$. Show that

$$\rho^2(p,q) \leq 2\eta(p,q) \leq 2\rho(p,q).$$

2. Define $K(q,p) = \int p(x) \log(\frac{p(x)}{q(x)}) dx$. Show that $K(q,p) \geq 2(1 - \rho(p,q))$.

3. Suppose $p(x)$ is the density function of $N(\mu_1, 1)$ distribution and $q(x)$ is the density of $N(\mu_2, 1)$ distribution. Calculate $\rho(p,q)$.

4. Suppose we observe sample X with density function $f(x)$. Consider the following simple Hypotheses testing problem:

$$H_0: f = f_0, \qquad H_1: f = f_1.$$

   Suppose we have prior distribution π given by $\pi(f_0) = p, \pi(f_1) = 1 - p$. Under the 0-1 loss, find the Bayes decision.

5. Suppose we observe sample X with density function $f(x)$. Now suppose we have multiple hypotheses $H_i: f = f_i$ for $i = 1,2, \dots, k$. The prior distribution π given by $\pi(f_i) = p_i$, where $p_i > 0$ and $\sum p_i = 1$. Under the 0-1 loss, find the Bayes decision.