Decoupling seminar – problem set 1

Here are a few problems to think about to help digest the material in the decoupling seminar.

We first recall the setup. We write $P \subset \mathbb{R}^n$ for the truncated paraboloid: $\{\omega \in \mathbb{R}^n | \omega_n = \sum_{i=1}^{n-1} \omega_i^2, \omega_n \le 1\}$. We suppose that $\hat{f}$ is supported in $N_{1/R}P$. We divide $N_{1/R}P$ into disjoint essentially rectangular slabs θ of dimensions $R^{-1/2} \times ... \times R^{-1/2} \times R^{-1}$. We define f_θ by $\hat{f}_\theta = \chi_\theta \hat{f}$. In particular, $f = \sum_\theta f_\theta$.

The decoupling theorem says that for any ball B of radius R in $\mathbb{R}^n$, and any $2 \le p \le \bar{p} := \frac{2(n+1)}{n-1}$, we have

$$\|f\|_{L^p(B)} \lesssim R^\epsilon \left(\sum_\theta \|f_\theta\|^2_{L^p(w_B)} \right)^{1/2}. \qquad (*)$$

Here w_B is a measure that is comparable to the standard Lebesgue measure on B and decays at a fast polynomial rate away from B.

1. Compute all the relevant norms in the following example. We let B be the ball of radius R centered at 0. For each θ, let η_θ be a smooth bump supported in θ of height 1, and let f_θ be the inverse Fourier transform $\eta_\theta^\vee$. Let $f = \sum_\theta f_\theta$. Compute the left-hand side and right-hand side of $(*)$, and check that in this example $(*)$ holds if and only if $2 \le p \le \bar{p} := \frac{2(n+1)}{n-1}$.

(If it's hard to make a completely rigorous proof, a heuristic argument for these norms is also very useful.)

2. The inequality $(*)$ implies that we also have decoupling on larger balls. More generally, suppose that $f = \sum f_i$, and that a domain A is a disjoint union of subsets A_j. On each A_j, suppose we have the inequality

$$\|f\|_{L^p(A_j)} \le M \left(\sum_i \|f_i\|^2_{L^p(A_j)} \right)^{1/2}.$$

Prove that the same inequality holds on all of A:

$$\|f\|_{L^p(A)} \le M \left(\sum_i \|f_i\|^2_{L^p(A)} \right)^{1/2}.$$

(Hint: Use the Minkowski inequality.)

3. There are some connections between periodic Strichartz and number theory – or more generally between decoupling and number theory. We outline an example based on eigenfunctions of the Laplacian.

Suppose that $\Lambda \subset \mathbb{R}^d$ is a lattice and that $T = \mathbb{R}^d/\Lambda$ is a flat torus. (The integer lattice is an interesting example, and other lattices are interesting too.) Suppose that g is an eigenfunction of the Laplacian on the torus T with eigenvalue λ. Then the solution of the Schrodinger equation $\partial_t u(x,t) = -i\triangle u(x,t)$ with initial data g is $u(x,t) = e^{i\lambda t} g(x)$. The initial data g has "frequency at most $\lambda^{1/2}$". By Bourgain-Demeter's Strichartz inequality, we get the following inequality, with $\bar{p} = \frac{2(d+2)}{d}$:

$$\|u\|_{L^{\bar{p}}(T\times[0,1])} \lesssim \lambda^{\epsilon} \|u(\cdot,0)\|_{L^2(T)}.$$

Because of the particular form of u, we get the following corollary about the eigenvalue g:

$$\|g\|_{L^{\bar{p}}(T)} \lesssim \lambda^{\epsilon} \|g\|_{L^2(T)}. \tag{1}$$

Recall that any function f on T can be expanded in a Fourier series $f(x) = \sum_{\omega \in 2\pi\Lambda^*} \hat{f}(\omega) e^{i\omega x}$. Now the eigenfunctions of the Laplacian on T with eigenvalue Λ are spanned by the complex exponentials $e^{i\omega x}$ with $\omega \in 2\pi\Lambda^*$ and with $|\omega|^2 = \lambda$. Let E_λ be the eigenspace with eigenvalue λ for the Laplacian on T. The bound (1) leads to an estimate on the dimension of E_λ.

If D_λ is the dimension of E_λ, first prove that we can find an eigenfunction $g \in E_\lambda$ with $\|g\|_{L^2(T)} = 1$ and with $|g(x_0)| \gtrsim D_\lambda^{1/2}$ for some point x_0.

Then using elliptic theory, prove that the average value of $|g|$ on $B(x_0, \lambda^{-1/2})$ is $\gtrsim D_\lambda^{1/2}$.

This last estimate gives a lower bound on $\|g\|_{L^{\bar{p}}(T)}$, and combining with equation (1) then gives an upper bound on D_λ.

As we mentioned above, the dimension D_λ is the number of lattice points of $2\pi\Lambda^*$ that lie on the sphere of radius $\lambda^{1/2}$. So this analysis leads to a non-trivial (but non-sharp) estimate for the number of lattice points on a sphere. For instance, it shows that the number of integer points on a sphere of radius r in $\mathbb{R}^3$ is $O(r^{1.8+\epsilon})$. A simple geometric argument shows an upper bound of $O(r^2)$, and I believe that the truth is $O(r^{1+\epsilon})$.