

Decoupling seminar – notes following lecture 3

For next week, I thought it would help to have some notes to read over about lecture 3. The goal of the notes is to give an outline of the proof of multilinear restriction from multilinear Keakeya. There are a lot of characters to keep track of. The first section of the notes is a statement of the theorems and the definitions to look over. I will hand out copies at the start of next lecture.

1. REVIEW OF THE THEOREMS AND THE CHARACTERS IN THE STORY

Theorem 1. (*Multilinear Keakeya*) Suppose that $l_{j,a}$ are lines in \mathbb{R}^n , with

$$\text{Angle}(l_{j,a}, x_j\text{-axis}) \leq (10n)^{-1}.$$

Let $T_{j,a,W}$ be the characteristic function of the W -neighborhood of $l_{j,a}$.

Let $g_j = \sum_a b_{j,a} T_{j,a,W}$ for some weights $b_{j,a} \geq 0$.

Then, for any cube Q_S of side length $S \geq W$, we have the following inequality:

$$(1) \quad \int_{Q_S} \prod_{j=1}^n g_j^{\frac{1}{n-1}} \lesssim (S/W)^\epsilon \prod_{j=1}^n \left(\int_{Q_S} g_j \right)^{\frac{1}{n-1}}.$$

Theorem 2. (*Multilinear restriction*) Suppose that $S_j \subset \mathbb{R}^n$ are C^2 compact hypersurfaces. Suppose that for any point $\omega \in S_j$, the normal vector $N_{S_j}(\omega)$ obeys

$$\text{Angle}(N_{S_j}(\omega), x_j\text{-axis}) \leq (10n)^{-1}.$$

Suppose that $\text{supp } \hat{f}_j \subset N_{1/R} S_j$.

Let μ_{B_R} denote a weight function which is 1 on B_R and decays rapidly away from B_R . Then

$$(2) \quad \int_{B_R} \prod_{j=1}^n |f_j|^{\frac{2}{n-1}} \lesssim R^\epsilon \prod_{j=1}^n \left(\int |f_j|^2 \mu_{B_R} \right)^{\frac{1}{n-1}}.$$

We decompose $N_{1/R} S_j$ as a disjoint union of blocks θ , each with dimensions $\sim R^{-1/2} \times \dots \times R^{-1/2} \times R^{-1}$. We define $f_{j,\theta}$ by $\hat{f}_{j,\theta} = \chi_\theta \hat{f}_j$, so that $f_j = \sum_\theta f_{j,\theta}$. We let ω_θ be the center of θ .

We let θ^* denote the dual tube, which has dimensions $\sim R^{1/2} \times \dots \times R^{1/2} \times R$. It is defined as follows:

$$\theta^* := \{x \in \mathbb{R}^n \text{ so that } |x \cdot (\omega - \omega_\theta)| \leq 1 \text{ for all } \omega \in \theta\}.$$

2. PROPERTIES OF $f_{j,\theta}$

In this section we give some heuristics about the properties of $f_{j,\theta}$. We state two “white lies” about the properties of the functions $f_{j,\theta}$. These white lies are not actually true, but I think that they are morally true. Using these properties, the “proof” of multilinear restriction is short and clean. Here are the two properties.

- (1) **Locally constant.** Morally, $|f_{j,\theta}|$ should be roughly constant on any translate of θ^* . More precisely, for any $x \in \mathbb{R}^n$,

$$\frac{\max_{y \in \theta^*} |f_{j,\theta}(x+y)|}{\min_{y \in \theta^*} |f_{j,\theta}(x+y)|} \leq C_n.$$

- (2) **Orthogonality.** If $B \subset \mathbb{R}^n$ is a ball of radius at least $R^{1/2}$ or a cube of side length at least $R^{1/2}$, then

$$\int_B |f_j|^2 \sim \sum_{\theta} \int_B |f_{j,\theta}|^2.$$

Neither of these two properties is actually true. I think of them as being morally true. In the next lecture, we will state more complicated results that are honestly true and that have the same spirit as the locally constant property and the orthogonality property. When we do that, the weight μ_{B_R} will enter the story.

3. THE “PROOF” OF MULTILINEAR RESTRICTION

In this section, we give a ‘proof’ of multilinear restriction assuming that the functions $f_{j,\theta}$ obeys the locally constant property and the orthogonality property. These properties are white lies, so this isn’t a completely rigorous proof, but it shows the main ideas of the real proof.

We define $M(R)$ to be the best constant in the inequality

$$(3) \quad \int_{Q_R} \prod_{j=1}^n |f_j|^{\frac{2}{n-1}} \leq M(R) \prod_{j=1}^n \left(\int_{Q_R} |f_j|^2 \right)^{\frac{1}{n-1}}.$$

(If the locally constant property and the orthogonality property were literally true, we would need a weight μ_{B_R} on the right-hand side.)

We want to prove that $M(R) \lesssim R^\epsilon$. We will prove an inequality that compares $M(R)$ and $M(R^{1/2})$, controlling the growth of $M(R)$. This inequality implies that $M(R)$ grows at most like R^ϵ .

We break Q_R into a disjoint union of smaller cubes of side length $R^{1/2}$. We evaluate the integral on each smaller cube, bringing in $M(R^{1/2})$. Using orthogonality and the locally constant property we see that the resulting integral looks like the LHS of the

multilinear Kakeya inequality. To help follow the argument, we indicate at the side when we use the locally constant property (LC), and when we use orthogonality (O).

$$\begin{aligned} \oint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} &= \text{Avg}_{Q_{R^{1/2}} \subset Q_R} \oint_{Q_{R^{1/2}}} \prod_j |f_j|^{\frac{2}{n-1}} \leq \\ &\leq \text{Avg}_{Q_{R^{1/2}} \subset Q_R} M(R^{1/2}) \prod_j \left(\oint_{Q_{R^{1/2}}} |f_j|^2 \right)^{\frac{1}{n-1}} \lesssim \end{aligned} \quad (O)$$

$$\lesssim \text{Avg}_{Q_{R^{1/2}} \subset Q_R} M(R^{1/2}) \prod_j \left(\oint_{Q_{R^{1/2}}} \sum_{\theta} |f_{j,\theta}|^2 \right)^{\frac{1}{n-1}} \lesssim \quad (LC)$$

$$\begin{aligned} &\lesssim \text{Avg}_{Q_{R^{1/2}} \subset Q_R} M(R^{1/2}) \oint_{Q_{R^{1/2}}} \prod_j \left(\sum_{\theta} |f_{j,\theta}|^2 \right)^{\frac{1}{n-1}} = \\ &= M(R^{1/2}) \oint_{Q_R} \prod_j \left(\sum_{\theta} |f_{j,\theta}|^2 \right)^{\frac{1}{n-1}}. \end{aligned}$$

By the locally constant property, the function $\sum_{\theta} |f_{j,\theta}|^2$ is essentially a weighted sum of characteristic functions of tubes that point in roughly the x_j direction. In other words, there is a weighted sum of tubes, g_j , so that $\sum_{\theta} |f_{j,\theta}|^2 \leq g_j \leq C_n \sum_{\theta} |f_{j,\theta}|^2$, and the functions g_j obey the hypotheses of the multilinear Kakeya inequality. Using the multilinear Kakeya inequality, we see that the last line is bounded by

$$\begin{aligned} &\lesssim M(R^{1/2}) R^{\epsilon} \prod_j \left(\oint_{Q_R} \sum_{\theta} |f_{j,\theta}|^2 \right)^{\frac{1}{n-1}} \lesssim \quad (O) \\ &\lesssim M(R^{1/2}) R^{\epsilon} \prod_j \left(\oint_{Q_R} |f_j|^2 \right)^{\frac{1}{n-1}}. \end{aligned}$$

All together, we see that

$$M(R) \leq C_{n,\epsilon} R^{\epsilon} M(R^{1/2}) \quad (*)$$

Iterating inequality (*) $\log \log R$ times shows that $M(R) \lesssim R^{\epsilon} M(10)$.