Decoupling seminar – problem set after lecture 4

This problem set is about the technical aspects of the proof of multilinear restriction that we started to discuss in lecture. The goal is to teach some techniques in the area that can be used to patch all of the white lies in our proof, and in similar arguments.

The tools have to do with localizing functions in physical space by using rapidly decaying cutoffs.

Section 1 sets up a little notation. Section 2, 3, 4, and 5 describe four different 'tricks of the trade'. Section 6 gives a short outline of how to repair the white-lies version of the proof of multilinear restriction using these techniques. Section 7 discusses a small technical strengthening of multilinear restriction which is useful in later applications.

1. Cutoff functions for convex sets

Consider μ a rapidly decaying positive function on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} \mu(x)dx = 1$. We note that on the unit ball $\mu(x) \gtrsim 1$ and μ is rapidly decaying elsewhere. Therefore, we can think of $\int |f|\mu$ as similar to $\mathrm{Avg}_{B_1} |f|$, but also including a rapidly decaying tail.

Given any such μ and a bounded open convex set A, we can build a cutoff function adapted to A by changing coordinates. There is an affine function $h : \mathbb{R}^n \to \mathbb{R}^n$ so that $B(c_n) \subset h(A) \subset B(C_n)$. The affine function has the form $h(x) = L(x) + v$, where L is a linear map and $v \in \mathbb{R}^n$. We define the cutoff function μ_A by

$$\mu_A(x) = \mathrm{Det}(L)\mu(h(x)).$$

We see that $\int \mu_A = 1$, and that $\mu_A(x) \sim |A|^{-1}$ for $x \in A$ and rapidly decaying as x moves away from A. We can think of $\int |f|\mu_A$ as similar to $\mathrm{Avg}_A |f|$, but also including a rapidly decaying tail.

2. Orthogonality

In class we proved the following orthogonality result.

Proposition 2.1. *There is a rapidly decaying μ so that the following holds. If g_α have $\operatorname{supp} \hat{g}_\alpha$ in disjoint cubes of side length ρ, and if Q is a cube of side length $\geq \rho^{-1}$, then*

$$\fint_Q |\sum_\alpha g_\alpha|^2 \lesssim \sum \int |g_\alpha|^2 \mu_Q.$$

Exercise. Using this result as a black box, prove that for any rapidly decaying μ, there is a rapidly decaying μ^+ so that the following holds:

$$\int |\sum_\alpha g_\alpha|^2 \mu_Q \lesssim \sum \int |g_\alpha|^2 \mu_Q^+.$$

Let $K \subset \mathbb{R}^n$ be a (bounded open) convex set with center of mass ω_K. (We think of K as living in Fourier space.) Define the dual convex body K^* by

$$K^* := \{x \in \mathbb{R}^n \text{ so that } |x \cdot (\omega_K - \omega)| \leq 1 \text{ for all } \omega \in K\}.$$

Exercise 2. Modifying the proof of Proposition 2.1, prove the following:

Proposition 2.2. *There is a rapidly decaying μ so that the following holds. If g_α have* $\operatorname{supp} \hat{g}_\alpha$ *in disjoint translates of a convex set K, and if A is a translate of K^*, then*

$$\fint_A |\sum_\alpha g_\alpha|^2 \lesssim \sum \int |g_\alpha|^2 \mu_A.$$

3. Locally constant properties

In class, we prove the following 'morally locally constant' estimate.

Proposition 3.1. *There is a rapidly decaying μ so that the following holds. If* $\operatorname{supp} \hat{g} \subset K$ *(a bounded open convex set), then $|g| \lesssim |g| * \mu_{K^*}$.*

Moreover, for any $x \in \mathbb{R}^n$,

$$\max_{x+K^*} |g| \lesssim \min_{x+K^*} |g| * \mu_{K^*}.$$

Exercise 1. As a corollary of this Proposition, prove a weak multilinear restriction estimate with a polynomial error factor: If f_j obey the hypotheses of multilinear restriction, then

$$\fint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} \lesssim R^{C_n} \prod_j \left(\int |f_j|^2 \mu_{Q_R} \right)^{\frac{1}{n-1}}.$$

Exercise 2. For any rapidly decaying μ, there is a rapidly decaying μ^+ so that the following hold. If $\operatorname{supp} \hat{g} \subset K$ (a bounded open convex set), then for any $x \in \mathbb{R}^n$,

$$\max_{x+K^*} |g| * \mu_{K^*} \lesssim \min_{x+K^*} |g| * \mu^+_{K^*}.$$

4. Two kinds of fudges

In the white lies version of the proof, we often have a quantity like $\fint_{Q_{R^{1/2}}} |f_{j,\theta}|^2$ which will need to be replaced by something involving a rapidly decaying cutoff $\mu_{Q_{R^{1/2}}}$. In the discussion so far, we've seen two different ways this cutoff function could appear:

$$\int |f_{j,\theta}|^2 \mu_{Q_{R^{1/2}}} \text{ or } \fint_{Q_{R^{1/2}}} (|f_{j,\theta}| * \mu_{Q_{R^{1/2}}})^2.$$

If we're not careful these different kinds of fudges can pile on top of each other in a confusing way. A useful technical point is that these two kinds of fudges are basically equivalent. Prove the following two lemmas that makes this precise.

Lemma 1. *For any rapidly decaying μ, there is a rapidly decaying μ^+ so that the following holds. If* $\operatorname{supp} \hat{g}$ *lies in a bounded open convex set K, and if A is a translate of K^*, then for any $p \geq 1$,*

$$\int |g|^p \mu_A \lesssim \fint_A (|g| * \mu_A^+)^p.$$

Lemma 2. *For any rapidly decaying μ, there is a rapidly decaying μ^+ so that the following holds. If* $\operatorname{supp} \hat{g}$ *lies in a bounded open convex set K, and if A is a translate of K^*, then for any $p \geq 1$,*

$$\fint_A (|g| * \mu_A)^p \lesssim \int |g|^p \mu_A^+.$$

5. A last localization trick

Here is one other issue that will come up when trying to remove the white lies from the proof of multilinear restriction. In the white lie version of orthogonality, we said that if g_α have $\operatorname{supp} \hat{g}_\alpha$ in disjoint cubes of side length ρ, and if Q is a cube of side length $\geq \rho^{-1}$, then

$$\fint_Q |\sum_\alpha g_\alpha|^2 \sim \sum \fint_Q |g_\alpha|^2. \qquad \text{(white lie 1)}$$

We proved a true estimate with a rapidly decaying cutoff, but it only goes in one direction:

$$\fint_Q |\sum_\alpha g_\alpha|^2 \lesssim \sum \int |g_\alpha|^2 \mu_Q.$$

Can we go the other way?

Exercise: Suppose that g_α have $\operatorname{supp} \hat{g}_\alpha$ in disjoint cubes of side length ρ, and Q is a cube of side length $\geq \rho^{-1}$. Show that it is not always true that

$$\sum \fint_Q |g_\alpha|^2 \lesssim \int |g|^2 \mu_Q.$$

To avoid using orthogonality "in the wrong direction", we need one last localization trick.

Exercise. Suppose we know that for any f_j with $\operatorname{supp} \hat{f}_j \subset N_{C/R} S_j$ as in the statement of multilinear restriction, that we have the following inequality:

$$\fint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} \leq M \prod_j \left(R^{-n} \int_{\mathbb{R}^n} |f_j|^2 \right)^{\frac{1}{n-1}}. \tag{global}$$

Using this as a black box, show that there is a rapidly decaying μ so that

$$\fint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} \leq C_n M \prod_j \left(\int |f_j|^2 \mu_{Q_R} \right)^{\frac{1}{n-1}}. \tag{localized}$$

Hint: Multiply f_j by a cutoff function ψ_{Q_R} with $\psi_{Q_R} \sim 1$ on Q_R and rapidly decaying, and with $\operatorname{supp} \hat{\psi}_{Q_R} \subset B(1/R)$.

Because of this exercise, the localized and global versions of the multilinear restriction inequality are essentially equivalent.

6. A rigorous proof of multilinear restriction

Suppose that for some rapidly decaying μ, we know that

$$\fint_{Q_{R^{1/2}}} \prod_j |f_j|^{\frac{2}{n-1}} \leq M(R^{1/2}) \prod_j \left(\int |f_j|^2 \mu_{Q_{R^{1/2}}} \right)^{\frac{1}{n-1}}.$$

Following the white lie proof of multilinear restriction, and using the tricks in the sections above, prove that

$$\fint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} \lesssim R^\epsilon M(R^{1/2}) \prod_j \left(R^{-n} \int_{\mathbb{R}^n} |f_j|^2 \right)^{\frac{1}{n-1}}.$$

Finally, using the last localization trick, conclude that there is some rapidly decaying μ^+ so that

$$\fint_{Q_R} \prod_j |f_j|^{\frac{2}{n-1}} \lesssim R^\epsilon M(R^{1/2}) \prod_j \left(\int |f_j|^2 \mu^+_{Q_R} \right)^{\frac{1}{n-1}}.$$

(With a little more thought, one can also arrange that $\mu^+ = \mu$, but it's not important.)

Iterating this argument, prove multilinear restriction. (It may be helpful to use Exercise 3.1 from Section 3 above as a base case for your iteration.)

7. A slightly stronger version of multilinear restriction

Show that the same arguments actually give the following technically slightly stronger result, which is convenient for the applications in decoupling:

Theorem 1. *There is a rapidly decaying μ so that the following holds. Let f_j be as in multilinear restriction. Then*

$$\mathrm{Avg}_{Q_1 \subset Q_R} \prod_j \left(\max_{Q_1} |f_j| \right)^{\frac{2}{n-1}} \lesssim R^\epsilon \prod_j \left(\int |f_j|^2 \mu_{Q_R} \right)^{\frac{1}{n-1}}.$$

Morally, the functions $|f_j|$ are locally constant at scale 1, and so this should be equivalent to multilinear restriction. With the technical tools here, we can prove it with essentially the same argument that proved multilinear restriction.