

## Decoupling seminar – problem set 2

Here are a few problems to think about to help digest the material in the decoupling seminar. In the second lecture, we discussed the multilinear Kakeya inequality, which depends crucially on the Loomis-Whitney inequality. In this pset, we recall the Loomis-Whitney inequality. You will study the proof and some applications.

We first recall the setup. Suppose that  $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  is the map that forgets the  $j^{th}$  coordinate:

$$\pi_j(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n).$$

**Theorem 1.** (*Loomis-Whitney*) If  $f_1, \dots, f_n \in L^1(\mathbb{R}^{n-1})$ , then

$$\int_{\mathbb{R}^n} \prod_{j=1}^n f_j(\pi_j(x)) \leq \prod_{j=1}^n \|f_j\|_{L^1(\mathbb{R}^{n-1})}.$$

1. The Loomis-Whitney inequality quickly implies the multilinear Kakeya inequality for axis-parallel tubes. Suppose that  $l_{j,a}$  are lines in  $\mathbb{R}^n$  parallel to the  $x_j$ -axis. Let  $T_{j,a,W}$  be the characteristic function of the  $W$ -neighborhood of  $l_{j,a}$ . Prove using Loomis-Whitney that

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left( \sum_{a=1}^{N_j} T_{j,a,W} \right)^{\frac{1}{n-1}} \leq C_n W^n \prod_{j=1}^n N_j^{\frac{1}{n-1}}.$$

2. Prove the Loomis-Whitney inequality. The proof involves using the Holder inequality in a clever way, and also Fubini.

a.) Start by proving the inequality in case  $n = 2$ . This case is quite easy.

b.) Then prove the inequality in dimension  $n$  using induction on  $n$ . In other words, at a certain moment in the proof, use the  $(n - 1)$ -dimensional version of the inequality.

You may want to focus on  $n = 3$  first. All the key ideas are already present in three dimensions.

3. The Loomis-Whitney inequality is closely related to the Sobolev inequality. In fact, LW quickly implies a slightly stronger version of the Sobolev inequality, which I think is called the Gagliardo-Nirenberg inequality.

**Theorem 2.** If  $f \in C_{comp}^1(\mathbb{R}^n)$ , then

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{j=1}^n \|\partial_j f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}.$$

**Corollary 3.** (*Sobolev inequality*) If  $f \in C_{comp}^1(\mathbb{R}^n)$ , then

$$\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

Prove Theorem 2 using the Loomis-Whitney inequality.

Hint: Let  $f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) = \int_{\mathbb{R}} |\partial_j f(x_1, \dots, x_n)| dx_j$ .

4. The Loomis-Whitney inequality has some geometric corollaries.

a.) If  $U \subset \mathbb{R}^n$  is an open set, and  $\pi_j(U)$  has  $(n-1)$ -dimensional volume less than 1 for every  $j$ , then  $U$  has  $n$ -dimensional volume less than 1.

This in turn implies the isoperimetric inequality with a non-sharp constant:

b.) If  $U \subset \mathbb{R}^n$  is a bounded open set with a smooth boundary, then

$$\text{Vol}_n(U) \leq C_n \text{Vol}_{n-1}(\partial U)^{\frac{n}{n-1}}.$$

In our applications, we don't need the sharp constant in the Loomis-Whitney inequality. I would be happy with the following type of estimate: If  $U \subset \mathbb{R}^3$  is an open set and the area of  $\pi_j(U)$  is at most 1 for each  $j$ , then  $\text{Vol } U \leq 10^9$ . This seems very intuitive to me. It says basically that if an object appears small when viewed from many angles, then the object is actually small. What is the simplest, least computational proof that you can find of this inequality?