THE MULTILINEAR KAKEYA INEQUALITY

1. The discussion from last time, heuristics and memories

Suppose that $\{T_i\}$ is a Kakeya set of tubes in $\mathbb{R}^n$. Each tube has radius 1 and length N, and there are $\sim N^{n-1}$ tubes. Suppose that $|\cup T_i| \sim N^{n-\gamma}$.

The number of unit cubes from the unit cube lattice that intersect $\cup T_i$ is $\sim N^{n-\gamma}$. We use the polynomial ham sandwich theorem to choose a polynomial P so that $Z(P)$ bisects each of these unit cubes. The degree of $Z(P)$ is $\lesssim N^{1-\gamma/n}$. What does such a polynomial tell us? Let $Q(K)$ be this set of cubes.

Consider one of the tubes, T_i. Let l be a line parallel to the axis of T_i, a randomly chosen parallel line in the tube T_i. For almost every choice of l, we have $|l \cap Z(P)| \leq deg(P) \lesssim N^{1-\gamma/n}$. On the other hand, the tube T_i contains $\sim N$ cubes of $Q(K)$, and $Z(P)$ bisects each of them. Let $Q(T_i)$ be this list of cubes. They are disjoint, so we get

$$Average_{q \in Q(T_i),l} |l \cap Z(P) \cap q| \lesssim N^{-1} deg(P) \lesssim N^{-\gamma/n}.$$

For a typical cube q, we know that $Z(P)$ bisects q, and yet $Average_l |Z(P) \cap q| \lesssim N^{-\gamma/n}$ is much smaller than 1. This is only possible if the surface $Z(P) \cap q$ is approximately parallel to the tube T_i. We can make a revised picture of the surface $Z(P)$ in the tube T_i.

So the geometry of the surface $Z(P)$ is connected with the geometry of the tubes T_i. If we try to imitate Dvir's proof of the finite field Nikodym or Kakeya conjectures, we are led to the following question. Extend each tube T_i a further length $\sim N$, and let q' be a unit cube in the extension. Is it true that $Z(P)$ approximately bisects q'? This type of question looks difficult, and it may be unlikely. The surface $Z(P)$ is approximately tangent to the tube T_i inside of T_i, but it's hard to know whether $Z(P)$ will bend sharply as soon as it leaves T_i and come nowhere near to q'.

I spent a while trying to force $Z(P)$ to hit q', and it was pretty frustrating. I would charge down one of the tubes T_i, trying to pin $Z(P)$ and carry it down to q', and $Z(P)$ would stay with me for a while and then swing out of the way, while I went charging harmlessly by...

However, the structure that we observed above does say something interesting about Kakeya sets. We noticed that for a typical cube $q \subset T_i$, the surface $Z(P)$ is approximately tangent to T_i. But there are many different tubes T_j containing q. With the method above, we can argue that T_j is approximately tangent to $Z(P)$ for most of the tubes. In fact, there must be a hyperplane $\pi(q)$, and the tubes T_j must usually be almost tangent to $\pi(q)$. This is a somewhat surprising structure, called planiness.

Planiness was first discovered by Katz, Laba, and Tao, in the paper "An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$." (Ann. of Math. (2) 152 (2000), no. 2, 383-446.) Planiness was one of the observations/tools that allowed them to prove that a Kakeya set of tubes in $\mathbb{R}^3$ (with mild additional hypotheses) has volume at least $N^{2.5+\epsilon}$. Later, Bennett, Carbery, and Tao proved stronger and more general planiness estimates in the paper "On the multilinear restriction and Kakeya conjectures" in Acta Math. 196 (2006), no. 2, 261-302. We will come to their work below.

If we had a hypothetical Kakeya set of tubes, a typical cube would lie in many tubes T_j. Without any experience, we might guess that the different tubes $T_j \supset q$ would point in a bush of directions that was pretty dense on the unit sphere. Suprisingly, they need to concentrate near to a plane. Another way to say this is that they don't form a whole lot of joints.

During the course, we met many theorems about the incidence patterns of lines in space. Each of these questions can be adapted to a question about long thin tubes instead of lines. Usually the adapted question is wide open. But for the joints problem, the adapted question has a nice answer based on the ideas we have just been discussing.

2. The generalized Loomis-Whitney inequality

We prove here an analogue of the joints theorem with long thin tubes instead of perfect lines.

Theorem 2.1. *(Bennett-Carbery-Tao, Guth) Suppose that $T_{j,a}$ are cylinders in $\mathbb{R}^n$ for $1 \le j \le n$ and $1 \le a \le A$. Each cylinder has radius 1 and infinite length. The axis of a cylinder $T_{j,a}$ makes an angle of $< (100n)^{-1}$ with the x_j-axis.*

Let I be the points which lie in one cylinder for each value of $j = 1...n$. In equations $I := \cap_{j=1}^n (\cup_{a=1}^A T_{j,a})$.

Then the volume of I is $\lesssim A^{\frac{n}{n-1}}$.

Remarks. If the tubes $T_{j,a}$ are parallel to the x_j-axis, then this estimate follows from the Loomis-Whitney inequality. We see that the projection of I to any coordinate hyperplane lies in A unit balls, and then Loomis-Whitney gives $|I| \lesssim A^{\frac{n}{n-1}}$. The case of axis-parallel cylinders is basically equivalent to the Loomis-Whitney inequality. The problem here is to see that the inequality remains true if we are allowed to tilt the tubes a few degrees.

History. BCT proved a tiny bit weaker estimate using monotonicity formulas for the heat equation. G proved this estimate using the polynomial method. This theorem can be thought of as a version of joints for nearly-orthogonal tubes. It implies, in particular, the joints theorem for nearly orthogonal lines.

The proof involves the idea of the directed volume of a surface. Suppose S is a smooth hypersurface in $\mathbb{R}^n$ with normal vector N. If v is a unit vector, we define the directed volume of S perpendicular to V by the formula

$$V_S(v) := \int_S |N \cdot v| dvol_S.$$

Notice that if the tangent plane of S is perpendicular to v, we have $|N \cdot v| = 1$, and if the tangent plane contains v, we have $|N \cdot v| = 0$. For example, we consider the directed volume of the unit circle in the direction $v = (0, 1)$. The directed volume of an arc of the upper semi-circle in direction v is exactly the change in the x-coordinate over the arc. Therefore, the directed volume of the whole upper semi circle is 2, and the directed volume of the whole circle is 4.

The computation for the circle generalizes as follows. Let π be the orthogonal projection from $\mathbb{R}^n$ to $v^\perp \subset \mathbb{R}^n$.

Lemma 2.2. $V_S(v) = \int_{v^\perp} |S \cap \pi^{-1}(y)| dvol(y)$.

As a corollary, we can immediately estimate the directed volume of a degree d variety in a cylinder T.

Lemma 2.3. *(Cylinder estimate) Let T be an infinite cylinder in $\mathbb{R}^{\ltimes}$ of radius r. Let v be a unit vector parallel to the axis of T. Let $Z(P)$ be the vanishing set of a polynomial P.*

Then $V_{Z(P) \cap T}(v) \lesssim r^{n-1} deg(P)$.

Proof. Let π be the projection from T to the cross-section $v^\perp \cap T$. This cross-section is just an (n-1)-dimensional ball of radius r. For almost every y in this ball, $|\pi^{-1}(y) \cap Z(P)| \le deg(P)$. By the last lemma, $V_{Z(P) \cap T}(v)$ is bounded by $deg(P)$ times the volume of the cross-section, which is $\sim r^{n-1}$. $\square$

Lemma 2.4. *If S is a hypersurface in $\mathbb{R}^n$, and $v_1, ..., v_n$ are unit vectors and the angle from v_j to the x_j-axis is $\le (100n)^{-1}$, then $Vol_{n-1} S \le 2 \sum_j V_S(v_j)$.*

Proof. At a given point of S with normal vector N, we have to prove that $\sum_j |N \cdot v_j| \ge 1/2$. If e_j are the coordinate vectors, then it's straightforward to check that $\sum_j |N \cdot v_j| \ge 1$ for any unit vector N. The vectors v_j are very close to e_j, and so the error has size $\le \sum_j |e_j - v_j| \le (1/100)$. $\square$

Now we can do the proof of the theorem.

Proof. Consider the unit cubical lattice. Let $Q_1, ..., Q_V$ be all the unit cubes in the lattice which intersect the set I. We will prove $V \lesssim A^{\frac{n}{n-1}}$.

Let P be a non-zero polynomial so that $Z(P)$ bisects each cube $Q_1, ..., Q_V$ and $degP \lesssim V^{1/n}$. This bisection requires a certain amount of area, therefore:

$$Vol_{n-1} Z(P) \cap Q_i \gtrsim 1.$$

Let $T_j(Q_i)$ be a tube from our list, in direction j, which intersects Q_i. Let $v_{j,i}$ be the direction of this tube. The directions $v_{1,i}, ..., v_{n,i}$ are nearly orthonormal, and so

$$\sum_{j=1}^{n} V_{Z(P)\cap Q_i}(v_{j,i}) \gtrsim Vol_{n-1} Z(P) \cap Q_i \gtrsim 1.$$

For each cube, choose one direction j so that $V_{Z(P)\cap Q_i} v_{j,i} \gtrsim 1$, and assign the cube Q_i to the tube $T_j(Q_i)$. We have V cubes and nA tubes, so one of the tubes has $\gtrsim V/A$ cubes assigned to it. Let T be this tube, and let v be its direction. We have $\gtrsim V/A$ cubes Q_i obeying the following conditions:

- The cube Q_i intersects T.
- $V_{Z(P)\cap Q_i}(v) \gtrsim 1$.

Let $\tilde{T}$ be a wider cylinder with radius $2n$ and with the same central axis as T. All of the cubes Q_i lie in $\tilde{T}$. Therefore, we have

$$V/A \lesssim V_{Z(P)\cap \tilde{T}}(v) \lesssim V^{1/n}.$$

The last inequality is by the cylinder estimate.

Rearranging we get $V \lesssim A^{\frac{n}{n-1}}$. □

3. Multilinear Kakeya

The strongest version of the Kakeya conjecture is the L^p version. If T_i are a Kakeya set of tubes of radius 1 and length N, the L^p Kakeya conjecture says that for each $\epsilon > 0$,

$$\int_{\mathbb{R}^n} |\sum_i \chi_{T_i}|^{\frac{n}{n-1}} \leq C_\epsilon N^\epsilon N^n. \tag{1}$$

Remarks: If we arrange the tubes in a disjoint way, the left hand side is $\sim N^n$. If we arrange them all centered at the origin, then the left hand side is $\sim (\log N) N^n$. If true, this conjecture gives essentially sharp bounds for $\| \sum_i \chi_{T_i} \|_p$ for every p. It implies that the union of tubes has volume at least $c_\epsilon N^{n-\epsilon}$ for any $\epsilon > 0$.

This conjecture is still wide open. The multilinear Kakeya conjecture allows us to control a positive fraction of all the terms - in a certain sense. First we rewrite the left hand side of (1).

$$\int |\sum_i \chi_{T_i}|^{\frac{n}{n-1}} = \int |\sum_i \chi_{T_i}|^{\frac{1}{n-1}} \cdot ... \cdot |\sum_i \chi_{T_i}|^{\frac{1}{n-1}}$$

On the right hand side we have a product of n identical copies of $|\sum_i \chi_{T_i}|^{\frac{1}{n-1}}$. Now we edit the formula, keeping only a constant fraction of the terms in each copy of $|\sum_i \chi_{T_i}|^{\frac{1}{n-1}}$. Let $I(j)$ be the subset of tubes T_i where the angle between $v(T_i)$ and the x_j axis is $\le (100n)^{-1}$. For each j, the number of such tubes is $\sim N^{n-1}$ - they form a positive fraction of all of the tubes.

Theorem 3.1. *(Bennett-Carbery-Tao) For any $\epsilon > 0$, there exists a constant C_ϵ so that for any Kakeya set of tubes,*

$$\int \prod_{j=1}^{n} |\sum_{i \in I(j)} \chi_{T_i}|^{\frac{1}{n-1}} \le C_\epsilon N^\epsilon N^n.$$

(In this inequality, the N^ϵ factor can actually be removed, see my paper "On the endpoint case of the Bennett-Carbery-Tao multilinear Kakeya inequality". But this takes a lot of extra work.)

This inequality is a generalization of the last theorem. We explain how they are related and we sketch the extra steps needed to prove the theorem. For any integers $\mu_1, ..., \mu_n \ge 0$, consider the set of points:

$$I(\mu) := \{x \in \mathbb{R}^n | 2^{\mu_j} \le |\sum_{i \in I(j)} \chi_{T_i}| < 2^{\mu_j + 1} \text{ for all } j.\}$$

The left hand side is

$$\sim \sum_\mu |I(\mu)| \prod_j 2^{\mu_j/(n-1)}.$$

Therefore, the theorem follows from the following lemma:

Lemma 3.2. *For each μ as above, $|I(\mu)| \lesssim N^n \prod_j 2^{-\mu_j/(n-1)}$.*

The lemma shows that each term in the sum above has size $\lesssim N^n$, and the number of terms is $\lesssim (\log N)^n$, and so we get a bound for the total of $\lesssim N^n (\log N)^n$, which proves the theorem.

If $\mu = 0$, we have $I(0)$ contained in the n-fold intersection set I defined above, and the inequality follows from the Theorem in the last section. The other values of μ are fairly similar.

Let us randomly choose $I'(j) \subset I(j)$, including each tube with probability $2^{-\mu_j}$. Let I' be the points lying in one tube T_i, $i \in I'(j)$ for each j. A point of $I(\mu)$ lies in I' with probability $\gtrsim 1$. With high probability, the size of $I'(j)$ is $\sim N^{n-1} 2^{-\mu_j}$. Therefore, our bound for $I(\mu)$ follows from the following lemma.

Lemma 3.3. *Let $T_{j,a}$ $a = 1 ... A_j$ be cylinders of radius 1 nearly parallel to the x_j axis. Then the volume of the set of points lying in at least one tube of each direction is $\lesssim \prod_{j=1}^n A_j^{\frac{1}{n-1}}$.*

If all the A_j happen to be equal, this lemma is exactly the theorem from the last section.

The case of unequal A_j requires an extra refinement in the proof. We cut each cube Q_i into many smaller pieces, and we choose P to bisect each smaller piece. The smaller pieces are arranged into a grid, cut more finely in the directions j where A_j is small and more coarsely in the directions where A_j is large. Details in the exercises...

(More details. Take a cube Q_i. Pick tubes $T_j(Q_i)$. Change coordinates so that the vectors $v(T_j(Q_i))$ become exactly orthogonal. In these coordinates, Q_i is not quite a cube, but contains a slightly smaller cube $\tilde{Q}_i$. Chop $\tilde{Q}_i$ into a grid, where the j^{th} direction is cut subdivided into $\prod_{j' \neq j} A_{j'}$. Choose $Z(P)$ to bisect each of these pieces. ...)

4. Sharp turns of algebraic varieties?

So far, the polynomial method has not led to any progress on the Kakeya problem. There are major difficulties in applying the methods we have seen to long thin tubes instead of perfect lines.

In the proof of finite field Kakeya or Nikodym, we use parameter counting to find a polynomial that vanishes in some places, and then we argue that the polynomial also must vanish somewhere else. This step plays a key role in most of the proofs we have seen in this course. It's hard to see whether something like this can work in the setting of tubes.

Suppose as in the first section that K is the union of a Kakeya set of $1 \times N$ tubes with surprisingly small volume, and that P is a polynomial so that $Z(P)$ bisects each cube of the unit lattice that intersects K. Pick a tube T from the Kakeya set, and imagine extending it to twice its length, and let q be a unit cube in this extension. Is there any hope that $Z(P)$ also roughly bisects q? We know that $Z(P)$ bisects all the cubes in T, and we've also seen that in most of these cubes $Z(P)$ is roughly parallel to T. If $Z(P)$ keeps going in the direction of its tangent plane, it will come reasonably close to q (although it's still not clear it will really hit q). But it's not at all clear whether $Z(P)$ will continue in the direction of its tangent plane. Perhaps $Z(P)$ will curve dramatically and go nowhere near q.

It might be helpful to understand better how many sharp bends there can be in a degree d algebraic surface. Here is a toy problem that gets at some of these issues.

Let P be a polynomial in two variables. Let $Pos(P) := \{x \in \mathbb{R}^2 | P(x) > 0\}$. For a given degree d, how closely can $Pos(P)$ look like the square $[-1,1]^2$? Recall that the Hausdorff distance from $Pos(P)$ to $[-1,1]^2$ is $< \epsilon$ if $[-1,1]^2$ lies in the ϵ-neighborhood of $Pos(P)$ and $Pos(P)$ lies in the ϵ-neighborhood of the square. Let $\epsilon(d)$ be the infimum over all degree d polynomials P of $dist_{Haus}(Pos(P), [-1,1]^2)$. Can we describe the order of magnitude of $\epsilon(d)$?

Very little is known about this. We know that $\epsilon(d) > 0$ for each d. The reason is that $dist_{Haus}(Pos(P), [-1,1]^2)$ varies lower semi-continuously as P moves in $V(d) \setminus \{0\}$. Multiplying P by a positive constant does not change $Pos(P)$, and so we can restrict attention to polynomials in the unit sphere of $V(d)$. By compactness the infimum is attained. But if $dist_{Haus}(Pos(P), [-1,1]^2)$ were zero, we would have $P = 0$ on the boundary of the square. Then P would vanish on the line $x = 1$, and $(x-1)$ would factor out of P. Write P as $(1-x)^a P_1(x,y)$, where $(1-x)$ does not divide P_1. The polynomial P_1 vanishes at only finitely many points of the line $x = 1$. If a is even, then we see that P_1 needs to vanish on the side of the square where $x = 1$, and then $1 - x$ divides P_1, and we get a contradiction. If a is odd, then we see that P_1 needs to vanish on the part of the line $x = 1$ where $|y| > 1$. This still implies that $1 - x$ divides P_1, and we get a contradiction.

If d is even, a nice example is the polynomial $P_d = 1 - x^d - y^d$. The set $Pos(P_d)$ is the unit ball in the L^d norm. As $d \to \infty$, it approaches the square, which is the unit ball in the L^∞ norm. For every even d, $Pos(P_d) \subset [-1,1]^2$, and $P_d > 0$ on the square $[-(1/2)^{1/d}, (1/2)^{1/d}]$. Now $1 - (1/2)^{1/d} \sim 1/d$, and so $dist_{Haus}(Pos(P_d), [-1,1]^2) \sim 1/d$. Hence $\epsilon(d) \lesssim 1/d$. It seems plausible that P_d is near-optimal and that $\epsilon(d) \gtrsim 1/d$.

The hard problem is to give quantitative lower bounds on $\epsilon(d)$. I don't know of any explicit lower bound in the literature. I worked on the problem, and I had a plan for a lower bound of the form e^{-e^d}...

I think the moral issue is to give quantitative bounds on how sharply a degree d curve can make a certain type of turn. It's important to keep in mind the following example. The zero set of the hyperbola $xy = \epsilon$ makes a very sharp turn near the origin, which looks something like the corner of a square. But the hyperbola has two branches, and so instead of being positive on approximately one quartant, it is positive on two opposite quartants, and its positive set does not really look like the neighborhood of a corner of a square. An algebraic curve can make an arbitrarily sharp turn if it looks locally like a hyperbola with two branches, but it is harder for it to make a sharp turn with only one branch.

I might have gone on too long about this toy problem. A solution to this problem would not directly lead to any bounds on Kakeya. Trying to go further with the polynomial method and tubes, this type of estimate seems to come up. In general, it might be helpful to have more quantitative estimates about the geometry of degree d algebraic surfaces.