HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

Consider a kernel $K_\alpha(x) := |x|^{-\alpha}$ and convolution $T_\alpha f := f * K_\alpha$. Last time, we looked at how T_α works when $f = \chi_{B_r}$ is the characteristic function on a ball of radius r.

Proposition 0.1. *$\|T_\alpha \chi_{B_r}\|_q \lesssim \|\chi_{B_r}\|_p$ if and only if $\alpha q > n$ and $n - \alpha + \frac{n}{q} = \frac{n}{p}$. Or equivalently, $p > 1$ and $\alpha = n(1 - \frac{1}{q} + \frac{1}{p})$.*

In fact, this result is true for general cases.

Theorem 0.2. *(Hardy-Littlewood-Sobolev) If $p > 1$ and $\alpha = n(1 - \frac{1}{q} + \frac{1}{p})$, then $\|T_\alpha f\|_q \lesssim \|f\|_p$.*

Apart from our previous examples, the next simplest example would be $f := \sum_j \chi_{B_j}$ where B_j are some balls. It is easy to treat nonoverlapping balls, but rather difficult in overlapping cases. So, it might be helpful to know about the geometry of overlapping balls.

1. Ball doubling

Lemma 1.1. *(Vitali Covering Lemma) If $\{B_i\}_{i \in I}$ is a finite collection of balls, then there exist a subcollection $J \subset I$ such that $\{B_j\}_{j \in J}$ are disjoint but $\bigcup_{i \in I} B_i \subset \bigcup_{j \in J} 3B_j$.*

What happens if I is infinite? It is no longer true for infinite I: consider $\{B(0,r) : r \in \mathbb{R}^+\}$. Any two of them are overlapping, so any disjoint subcollection can contain only one ball. You cannot cover whole space by a bounded ball, so the theorem is false for this case. How can we fix it? If we loosen the conclusion to cover only a compact set $K \subset \bigcup_{i \in I} B_i$, then we can always find a disjoint subcollection $J(K) \subset I$ such that $K \subset \bigcup_{j \in J(K)} 3B_j$.

From Vitali covering lemma, we get the following:

Lemma 1.2. *(Ball doubling) If $\{B_i\}_{i \in I}$ is a finite collection of balls, then $|\bigcup 2B_i| \leq 6^n |\bigcup B_i|$.*

Proof. From the proof of Vitali Covering Lemma, for each B_i we can find some $j \in J$ such that $B_i \subset 3B_j$. So, $2B_i \subset 6B_j$. Hence $|\bigcup 2B_i| \leq |\bigcup 6B_j| \leq 6^n \sum |B_j| = 6^n |\bigcup B_j|$. □

Is it sharp? It seems to be 2^n instead of 6^n, but I'm not sure and at least hard to prove. This coefficient is not so important for the proof be given later, so let's go over it.

2. Hardy-Littlewood maximal function

Denote the average of f on A by $\fint_A f := \frac{1}{\mathrm{Vol} A}\int_A f$. The *Hardy-Littlewood maximal function* of f is defined to be $Mf(x) := \sup_r \fint_{B(x,r)} |f|$. Let $S_g(h) := \{x \in \mathbb{R}^n : |g| > h\}$. Then,

Lemma 2.1. $|S_{Mf}(h)| \lesssim h^{-1}\|f\|_1$.

Proof. For each $x \in S_{Mf}(h)$, there exists $r(x)$ such that $\fint_{B(x,r(x))} |f| \geq h$, so $\int_{B(x,r(x))} |f| \geq h|B(x, r(x))|$. These $B(x, r(x))$ cover $S_{Mf}(h)$, so by Vitali covering lemma, we can find disjoint B_j's whose multiple cover $S_{Mf}(h)$. Hence,

$$|S_{Mf}(h)| \lesssim \sum_j |B_j| \lesssim h^{-1} \fint_{\bigcup B_j} |f| \leq h^{-1}\|f\|_1.$$

□

Now we can estimate the L_p-norm of Mf by that of f.

Proposition 2.2. $\|Mf\|_p \lesssim \|f\|_p$.

One naive approach would be dividing the range and estimate in each range. Namely, let $T_{Mf}(2^k) := \{x \in \mathbb{R}^n : 2^k < |Mf| \leq 2^{k+1}\} \subset S_{Mf}(2^k)$ and we have

$$\int |Mf|^p \sim \sum_{k=-\infty}^{\infty} |T_{Mf}(2^k)|2^{kp} \lesssim \sum_k 2^{-k}2^{kp}\|f\|_1,$$

but the summation in the righthand side diverges. We need a slight modification of the previous lemma.

Lemma 2.3. $|S_{Mf}(h)| \lesssim h^{-1}\int_{S_f(h/2)} |f|$.

Proof. In the previous proof, we found disjoint B_j which covering $S_{Mf}(h)$ such that $\int_{B_j} |f| \geq h|B_j|$. However, we also have $\int_{B_j \setminus S_f(h/2)} |f| \leq \frac{h}{2}|B_j|$, so $\int_{B_j \cap S_f(h/2)} |f| \geq \frac{h}{2}|B_j|$. Do the same estimate with $B_j \cap S_f(h/2)$ instead of B_j and get the desired result. □

Now we can prove the proposition.

Proof. Use the same approach above with our modified lemma.

$$\int |Mf|^p \lesssim \sum_{k=-\infty}^{\infty} |S_{Mf}(2^k)|2^{kp} \lesssim \sum_k 2^{k(p-1)} \int_{S_f(2^{k-1})} |f|.$$

By interchanging summation and integral, we have

$$\int |f| \sum_{2^{k-1} \leq |f|} 2^{k(p-1)} \sim \int |f| \cdot |f|^{p-1} = \|f\|_p^p.$$

So, $\|Mf\|_p \lesssim \|f\|_p$. ☐

3. Proof of HLS Inequality

Step 1. $T_\alpha f(x)$ can be written in terms of $\fint_{B(x,r)} f$.

Lemma 3.1.

$$T_\alpha f(x) = \int_0^\infty r^{n-\alpha-1} \left(\fint_{B(x,r)} f \right) dr.$$

Proof. Just a computation. ☐

Step 2. Upper bounds of $\fint_{B(x,r)} f$. One trivial upper bound is $Mf(x)$ by definition. Also, we can get

$$\fint_{B(x,r)} f \lesssim r^{-n} \int_{B(x,r)} |f| \lesssim r^{-n} \|f\|_p r^{n(p-1)/p} = r^{-n/p} \|f\|_p$$

by Hölder. We would fail if we only use one of them. Rather, fix $r_{crit}(x)$ and use $Mf(x)$ for $r \leq r_{crit}$, L^p bound for $r \geq r_{crit}$. This approximation always gives us $|T_\alpha f(x)| \lesssim (Mf)^A \|f\|_p^B$ for some A, B with $A + B = 1$.

Step 3. $\int |T_\alpha f|^q \lesssim \|f\|_p^{Bq} \int (Mf)^{Aq} \lesssim \|f\|_p^{Bq} \|f\|_{Aq}^{Aq}$ as long as $Aq > 1$. If $p = Aq$, then we have $\int |T_\alpha f|^q \lesssim \|f\|_p^q$, so $\|T_\alpha f\|_q \lesssim \|f\|_p$. This case together with $Aq > 1$ is exactly the hypothesis condition in the theorem. Also, we already know that this condition is the only possible case, so we are done. You may calculate r_{crit}, A, B to check.