USING (POLYNOMIAL) CELL DECOMPOSITIONS

1. Szemerédi-Trotter

We will recall the standard form of the theorem.

Theorem 1.1. *If $\mathfrak{S}$ is a set of S points and $\mathfrak{L}$ is a set of L lines (all in $\mathbb{R}^2$), then the number of incidences obeys the following bound:*

$$I(\mathfrak{S}, \mathfrak{L}) \leq C_0[S^{2/3}L^{2/3} + S + L].$$

We will prove the result by using a polynomial cell decomposition together with elementary counting bounds in each cell. We first recall the counting bounds.

Lemma 1.2. *If $\mathfrak{S}$ and $\mathfrak{L}$ are as above, then*

- $I(\mathfrak{S}, \mathfrak{L}) \leq L + S^2$.
- $I(\mathfrak{S}, \mathfrak{L}) \leq L^2 + S$.

Proof. Fix $x \in \mathfrak{S}$. Let L_x be the number of lines of $\mathfrak{L}$ that contain x and no other point of $\mathfrak{S}$. For each other point $y \in S$, there is at most one line of $\mathfrak{L}$ containing x and y. Therefore, $I(x, \mathfrak{L}) \leq S + L_x$. So $I(\mathfrak{S}, \mathfrak{L}) \leq S^2 + \sum_{x \in S} L_x \leq S^2 + L$.

The proof of the other inequality is similar. ∎

Now we turn to the proof of the theorem.

Proof. If $L > S^2/10$ or $S > L^2/10$, then the conclusion follows from the counting lemma. Therefore, we can now restrict to the case that

$$10^{1/2}S^{1/2} \leq L \leq S^2/10. \tag{1}$$

We will also use induction on L, and so we can assume the theorem holds for smaller sets of lines.

Now we come to the heart of the proof. We use the polynomial cell decomposition to cut $\mathbb{R}^2$ into cells, and then we use the counting lemma in each cell.

Let d be a degree to choose later. By the polynomial cell decomposition lemma, we can find a non-zero polynomial P of degree $\leq d$ so that each component of the complement of $Z(P)$ contains $\lesssim Sd^{-2}$ points of $\mathfrak{S}$. Let O_i be the components, S_i the number of points of $\mathfrak{S}$ in O_i, and L_i the number of lines of $\mathfrak{L}$ that intersect O_i. Since each line intersects $\leq d + 1$ cells, we know that $\sum L_i \leq L(d + 1)$.

Applying the counting lemma in each cell, we get

$$I(\mathfrak{S}_i, \mathfrak{L}_i) \le L_i + S_i^2.$$

We let $\mathfrak{S}_{cell}$ be the union of $\mathfrak{S}_i$ - all the points of $\mathfrak{S}$ that lie in the interiors of the cells.

$$I(\mathfrak{S}_{cell}, \mathfrak{L}) = \sum_i I(\mathfrak{S}_i, \mathfrak{L}_i) \le \sum_i L_i + \sum_i S_i^2 \lesssim Ld + Sd^{-2} \sum_i S_i = Ld + S^2 d^{-2}.$$

We let $\mathfrak{S} = \mathfrak{S}_{cell} \cup \mathfrak{S}_{alg}$, where $\mathfrak{S}_{alg}$ is the set of points in $Z(P)$. It remains to bound $I(\mathfrak{S}_{alg}, \mathfrak{L})$. We divide $\mathfrak{L}$ as $\mathfrak{L}_{cell} \cup \mathfrak{L}_{alg}$, where $\mathfrak{L}_{cell}$ are the lines that intersect some open cells, and $\mathfrak{L}_{alg}$ are the lines contained in $Z(P)$.

Each line of $\mathfrak{L}_{cell}$ has $\le d$ intersection points with $Z(P)$, hence $\le d$ incidences with $\mathfrak{S}_{alg}$. Hence $I(\mathfrak{S}_{alg}, \mathfrak{L}_{cell}) \le Ld$. Summarizing everything so far, we have the following:

$$I(\mathfrak{S}, \mathfrak{L}) \le C(Ld + S^2 d^{-2}) + I(\mathfrak{S}_{alg}, \mathfrak{L}_{alg}).$$

We will deal with the last term by induction. We will choose $d \le L/2$. So $\mathfrak{L}_{alg}$ contains $\le L/2$ lines. By induction,

$$I(\mathfrak{S}_{alg}, \mathfrak{L}_{alg}) \le C_0[S^{2/3}(L/2)^{2/3} + S + L/2].$$

Now we are ready to optimize over d. We need to choose d to be an integer between 1 and $L/2$. We choose $d \sim S^{2/3}L^{-1/3}$. Because of the bounds in equation (1), we can find d this size in the range $1 \le d \le L/2$. Plugging in, we get

$$I(\mathfrak{S}, \mathfrak{L}) \le CL^{2/3}S^{2/3} + C_0[S^{2/3}(L/2)^{2/3} + S + L/2].$$

Finally, we choose C_0 large enough compared to C, and the whole right hand side is bounded by $C_0[S^{2/3}L^{2/3} + S + L]$.

$\square$

2. The 3-dimensional version - outline of the ideas

We will prove (today and next lecture) the following 3-dimensional result, which we can think of as a possible analogue of the ST theorem for lines in $\mathbb{R}^3$.

Theorem 2.1. *If $\mathfrak{S}$ is a set of S points in $\mathbb{R}^3$ and $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with at most B lines in any plane, then*

$$I(\mathfrak{S}, \mathfrak{L}) \le C_0[S^{1/2}L^{3/4} + L^{1/3}B^{1/3}S^{2/3} + S + L].$$

In particular, if k is sufficiently large and we take $\mathfrak{S}$ to be the set of points in $\geq k$ lines of $\mathfrak{L}$, then plugging in we get $|\mathfrak{S}_k| \lesssim L^{3/2}k^{-2} + LBk^{-3} + Lk^{-1}$. Taking $B = L^{1/2}$ and combining with our earlier bound for 3-rich points, we get

Corollary 2.2. *If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with $\leq L^{1/2}$ lines in any plane and $k \geq 3$, then the number of k-rich points is $\lesssim L^{3/2}k^{-2}$.*

Now we discuss some examples. The S term and the L term are easy. If we choose L/B planes, and use a grid configuration in each plane, we get $\sim L^{1/3}B^{1/3}S^{2/3}$ incidences. Finally, if we choose points and lines coming from a 3-dimensional grid, we can get $S^{1/2}L^{3/4}$ incidences. In particular, the theorem is sharp up to constant factors.

The main ideas are similar to the ideas in the proof of ST above, but there are one or two extra twists and the computations are longer. In this outline, we want to explain the main steps/ideas, especially the new twists, but postpone the calculations.

We let d be a degree we can choose later, and we build a degree d polynomial cell decomposition. In each cell we apply an incidence bound that we already know. We could apply the counting lemma as above. We can also apply the Szemerédi-Trotter theorem in each cell. Recall that the Szemerédi-Trotter theorem holds for points and lines in $\mathbb{R}^n$ for any n by a random projection argument. Since it is stronger than the counting lemma bounds, we may as well use ST in each cell. Then adding up the contributions from the cells, we get

$$I(\mathfrak{S}_{cell}, \mathfrak{L}) \lesssim S^{2/3}L^{2/3}d^{-1/3} + S + L.$$

As d increases, we get stronger and stronger bounds on the incidences in the cells. On the other hand, as d increases, we get more points in $Z(P)$ and weaker information about $Z(P)$.

We can again divide the lines as $\mathfrak{L}_{cell}$ and $\mathfrak{L}_{alg}$. Each line of $\mathfrak{L}_{cell}$ has $\leq d$ incidences with $\mathfrak{S}_{alg}$. Therefore, we get

$$I(\mathfrak{S}, \mathfrak{L}) \lesssim dL + d^{-1/3}S^{2/3}L^{2/3} + S + L + I(\mathfrak{S}_{alg}, \mathfrak{L}_{alg}).$$

In the proof of ST, we chose $d \leq L/2$, which forced $L_{alg} \leq L/2$ and allowed us to use induction. We cannot quite do that here. A surface of low degree may contain arbitrarily many lines. This is true for planes and reguli, and also for many other examples. We cannot yet use induction. Also, we need to use the bound on the number of lines in a plane, which we haven't used yet.

The surface $Z(P)$ contains $\leq d$ planes. Each of these planes contains $\leq B$ lines of $\mathfrak{L}$. Let $\mathfrak{L}_{planar}$ be the subset of lines of $\mathfrak{L}$ which lie in one of the planes of $Z(P)$. Using this information and applying Szemerédi-Trotter in each plane, it's not hard to bound $I(\mathfrak{S}, \mathfrak{L}_{planar})$. In particular, we'll get the following bound:

$$I(\mathfrak{S}, \mathfrak{L}_{planar}) \lesssim B^{1/3} L^{1/3} S^{2/3} + dL + S + L.$$

This estimate is fine, and it remains to bound $I(\mathfrak{S}_{alg}, \mathfrak{L}_{alg} \setminus \mathfrak{L}_{planar})$. We will do this using our tools about special points and lines in an algebraic surface – as in the proof of the esimate on the number of 3-rich points. As in that lecture, we call a point special if it is critical or flat, and we call a line special if each point on the line is special. A point $x \in Z(P)$ is special if and only if a set of polynomials called SP vanishes at x, and the polynomials in SP have degree $\leq 3d$.

One of the main tools in the special lines discussion is that there aren't that many special lines. The number of special lines in $Z(P)$ which aren't in any of the planes is $\leq 10d^2$. We will choose d so that $10d^2 \leq L/2$, and then we can control this term by induction. We write $\mathfrak{L}_{alg} = \mathfrak{L}_{spec} \cup \mathfrak{L}_{nonspec}$ where $\mathfrak{L}_{spec}$ are the special lines of $\mathfrak{L}_{alg}$. Note that $\mathfrak{L}_{planar} \subset \mathfrak{L}_{spec}$. We just recalled that $|\mathfrak{L}_{spec} \setminus \mathfrak{L}_{planar}| \leq 10d^2$.

We have

$$I(\mathfrak{S}_{alg}, \mathfrak{L}_{alg} \setminus \mathfrak{L}_{planar}) \leq I(\mathfrak{S}_{alg}, \mathfrak{L}_{spec} \setminus \mathfrak{L}_{planar}) + I(\mathfrak{S}_{alg}, \mathfrak{L}_{nonspec}).$$

We can control the first term by induction as long as we choose $10d^2 \leq L/2$. And we will see that the second term is minor.

We write $\mathfrak{S}_{alg} = \mathfrak{S}_{spec} \cup \mathfrak{S}_{nonspec}$. Each non-special line contains at most $3d$ special points, so

$$I(\mathfrak{S}_{spec}, \mathfrak{L}_{nonspec}) \leq 3dL.$$

On the other hand, if a point $x \in Z(P)$ lies in three lines in $Z(P)$, then we saw that x is a special point of $Z(P)$. Therefore, each point of $\mathfrak{S}_{nonspec}$ is incident to $le2$ lines of $\mathfrak{L}_{alg}$. In particular, we get

$$I(\mathfrak{S}_{nonspec}, \mathfrak{L}_{nonspec}) \leq 2S.$$

Combining all the work so far, we see that

$$I(\mathfrak{S}, \mathfrak{L}) \leq C[dL + d^{-1/3} S^{2/3} L^{2/3} + B^{1/3} L^{1/3} S^{2/3} + S + L] + I(\mathfrak{S}_{alg}, \mathfrak{L}_{spec} \setminus \mathfrak{L}_{planar}).$$

This inequality holds for any integer $d \geq 1$, and if $10d^2 \leq L/2$, then the number of lines in $\mathfrak{L}_{spec} \setminus \mathfrak{L}_{planar}$ is $\leq L/2$, and we can control that term by induction. We optimize d in this range, and we get the bound in the theorem.

(In the full proof, we have to be a touch more careful about some of the terms because of the induction.)

Next lecture we will do the details.