Problem set for Math 158, unit on decoupling in Fourier analysis

1. Locally constant property. Suppose that $\theta \subset \mathbb{R}^2$ is a rectangle. Let $\theta^*$ denote the dual rectangle. If we rotate and translate so that $\theta = [0, \theta_1] \times [0, \theta_2]$, then the dual rectangle $\theta^*$ is $[0, \theta_1^{-1}], [0, \theta_2]^{-1}$. Suppose that the support of $\hat{f}$ is contained in $\theta$.

Prove that $|f|$ is roughly locally constant on translates of $\theta^*$ in the following sense: for any $x_0 \in \mathbb{R}^2$,

$$\|f\|_{L^\infty(\theta^*+x_0)} \lesssim \|f\|_{L^1_{avg}(W_{\theta^*+x_0})}.$$  

Here $W_{\theta^*+x_0}$ is a weight function which is $\sim 1$ on $\theta^* + x_0$ and decays rapidly away from $\theta^* + x_0$. If $\theta^*$ is $[0, \theta_1^{-1}], [0, \theta_2]^{-1}$, and $x_0 = 0$, then for any $N \geq 1$,

$$|W(y)| \lesssim N (1 + \theta_1 y_1 + \theta_2 y_2)^{-N}.$$  

For a weight $W$, the averaged norm $L^p_{\text{avg}}(W)$ is defined as

$$\|f\|_{L^p_{\text{avg}}(W)} := \left( \frac{\int W|f|^p}{\int W} \right)^{1/p}.$$  

2. The simplest decoupling problem. Let $\tau = [0, A] \subset \mathbb{R}$ and decompose $\tau = \cup \theta$. Define the decoupling constant $D_{\text{interval}}^p(A)$ to be the best constant in the following inequality:

Whenever $\hat{f}_\theta$ is supported in $\theta$ and $f = \sum_{\theta} f_\theta$, then

$$\|f\|_{L^p(\mathbb{R})} \leq D_{\text{interval}}^p(A) \left( \sum_{\theta} \|f_\theta\|_{L^p(\mathbb{R})}^2 \right)^{1/2}.$$  

Give the best upper and lower bounds that you can for $D_{\text{interval}}^p(A)$.

I might add one more problem next week.