Problem set for Math 158, unit on decoupling in Fourier analysis

1. Locally constant property. Suppose that $\theta \subset \mathbb{R}^2$ is a rectangle. Let θ^* denote the dual rectangle. If we rotate and translate so that $\theta = [0, \theta_1] \times [0, \theta_2]$, then the dual rectangle θ^* is $[0, \theta_1^{-1}], [0, \theta_2]^{-1}$. Suppose that the support of $\hat{f}$ is contained in θ.

Prove that $|f|$ is roughly locally constant on translates of θ^* in the following sense: for any $x_0 \in \mathbb{R}^2$,

$$\|f\|_{L^\infty(\theta^* + x_0)} \lesssim \|f\|_{L^1_{avg}(W_{\theta^* + x_0})}.$$

Here $W_{\theta^* + x_0}$ is a weight function which is ~ 1 on $\theta^* + x_0$ and decays rapidly away from $\theta^* + x_0$. If θ^* is $[0, \theta_1^{-1}], [0, \theta_2]^{-1}$, and $x_0 = 0$, then for any $N \geq 1$,

$$|W(y)| \lesssim_N (1 + \theta_1 y_1 + \theta_2 y_2)^{-N}.$$

For a weight W, the averaged norm $L^p_{avg}(W)$ is defined as

$$\|f\|_{L^p_{avg}(W)} := \left(\frac{\int W |f|^p}{\int W} \right)^{1/p}.$$

2. The simplest decoupling problem. Let $\tau = [0, A] \subset \mathbb{R}$ and decompose τ into unit intervals $\tau = \cup \theta$. Define the decoupling constant $D[interval]_p(A)$ to be the best constant in the following inequality:

Whenever $\hat{f}_\theta$ is supported in θ and $f = \sum_\theta f_\theta$, then

$$\|f\|_{L^p(\mathbb{R})} \leq D[interval]_p(A) \left(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R})} \right)^{1/2}.$$

Give the best upper and lower bounds that you can for $D[interval]_p(A)$.

I might add one more problem next week.