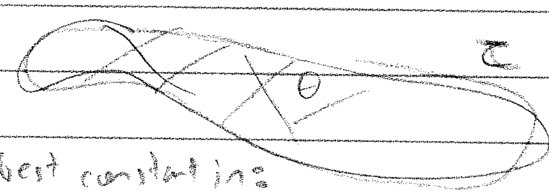
Decoupling Lecture 2

Review

$\tau \subset \mathbb{R}^n$



$\tau = \sqcup \theta$

$D_p(\tau = \sqcup\theta)$ is best constant in:

If $\quad \text{supp } \widehat{f_\theta} \subset \theta, \qquad f_\tau = \sum_{\theta \subset \tau} f_\theta$

then

$$\|f\|_{L^p(\mathbb{R}^n)} \le D_p(\tau = \sqcup\theta) \cdot \Big(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \Big)^{\frac{1}{2}}$$

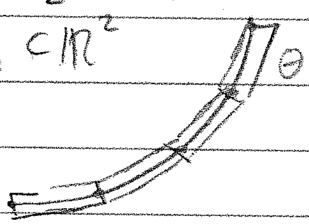
No complete theory, but a number of interesting cases.

Exer. 1 $\quad \tau = [0, A] = \bigsqcup^{A} [\cancel{t}-1, t]$ $\quad$ Decoupling as a gen. of Plancherel...

Rmk. $D_2(\quad) = 1 \quad t=1$ by Plancherel.

Parabola $\subset \mathbb{R}^2$

(P)

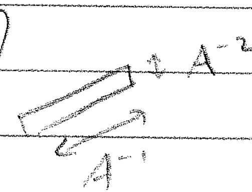


$T = \{ w_2 = w_1^2 \qquad 0 \le w_1 \le 1 \}$

cut into A arcs.

(thicken by A^{-2})

Each $\theta \approx$ rect.



Notation: $|\theta| = A^{-1}$

$D_p(A) :=$ decoupling for these θ.

THM. (Bourgain-Demeter '14) $\quad D_p(A) \lessapprox A^{\varepsilon}$.

for $\quad 2 \le p \le 6$

Decoupling on balls

Prop. If $f = \sum_{\substack{\theta \\ |\theta| = A^{-1}}} f_\theta$ as in (P)

then

$$\|f\|_{L^p(B_{A^2})} \lesssim D_p(A) \cdot \left(\sum_\theta \|f_\theta\|^2_{L^p(W_{B_{A^2}})} \right)^{\frac{1}{2}}$$

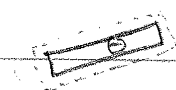
sketch pf. Choose η s.t.

$|\eta| \sim 1$ on B_{A^2}, rapidly decaying.

$\operatorname{supp} \hat{\eta} \subseteq B_{\frac{1}{100A^2}}$

$$\|f\|_{L^p(B_{A^2})} \lesssim \|\eta f\|_{L^p(\mathbb{R}^n)} \qquad \eta f = \sum_\theta \eta f_\theta$$

$$\widehat{\eta f_\theta} = \underbrace{\hat{\eta} * \hat{f_\theta}}_{\text{supported on}}$$



$\operatorname{supp} \hat{\eta} * \hat{f_\theta} \subseteq$

Geometry is almost same

$$\lesssim D_p(A) \cdot \left(\sum_\theta \|\eta f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}}$$

$$\leq D_p(A) \cdot \left(\sum_\theta \|f_\theta\|^2_{L^p(W_{B_{A^2}})} \right)^{\frac{1}{2}} . \qquad \square$$

Cor. $\displaystyle \|f\|_{L^2(B_{A^2})} \lesssim \left(\sum_\theta \|f_\theta\|^2_{L^2(W_{B_{A^2}})} \right)^{\frac{1}{2}}.$

Study individual f_θ

$\mathcal{F}(\Omega) := \{ f : \operatorname{supp} \hat{f} \subseteq \Omega \}$

Lem. If $f \in \mathcal{F}(B_1)$ then $\forall x_0$

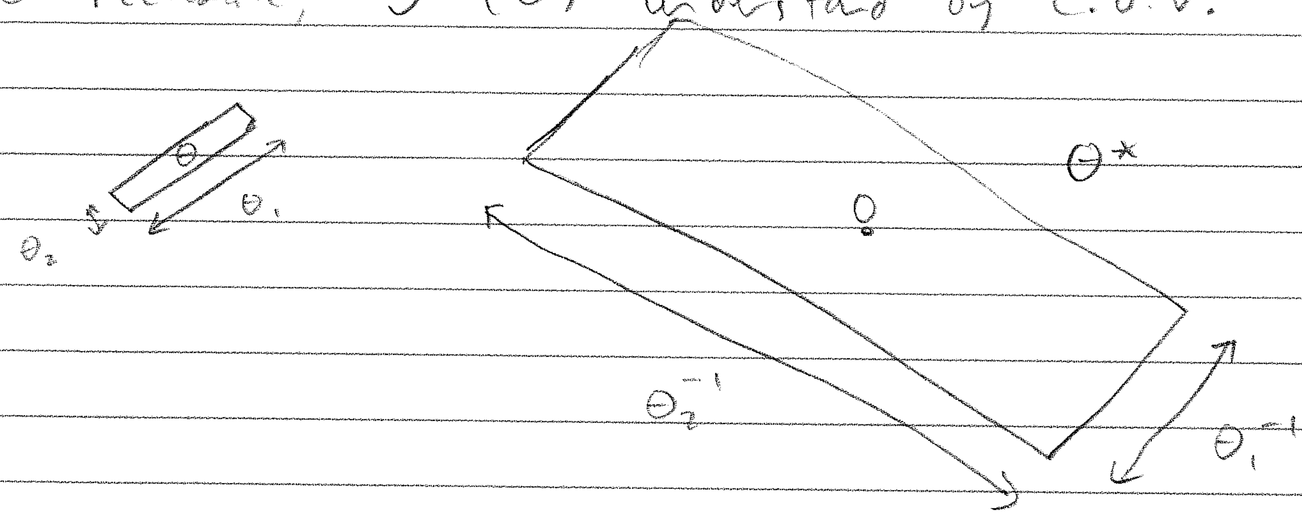
$$\|f\|_{L^\infty(B(x_0,1))} \lesssim \|f\|_{L^1(W_{B(x_0,1)})}. \quad \text{(LC)}$$

(Almost says $|f| \approx$ constant on $B(x_0,1)$.)

Rmk. $f \in \mathcal{F}(B(\omega_0,1)) \longleftrightarrow e^{-i\omega_0 x} f \in \mathcal{F}(B(0,1))$

Cor. If $f \in \mathcal{F}(B(\omega_0,1))$ then (LC).

If θ rectangle, $\mathcal{F}(\theta)$ understand by C.O.V.

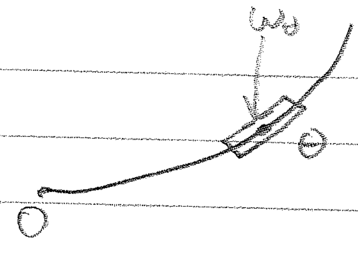


Lem. (Exercise) If $f \in \mathcal{F}(\theta)$, then $\forall x_0$

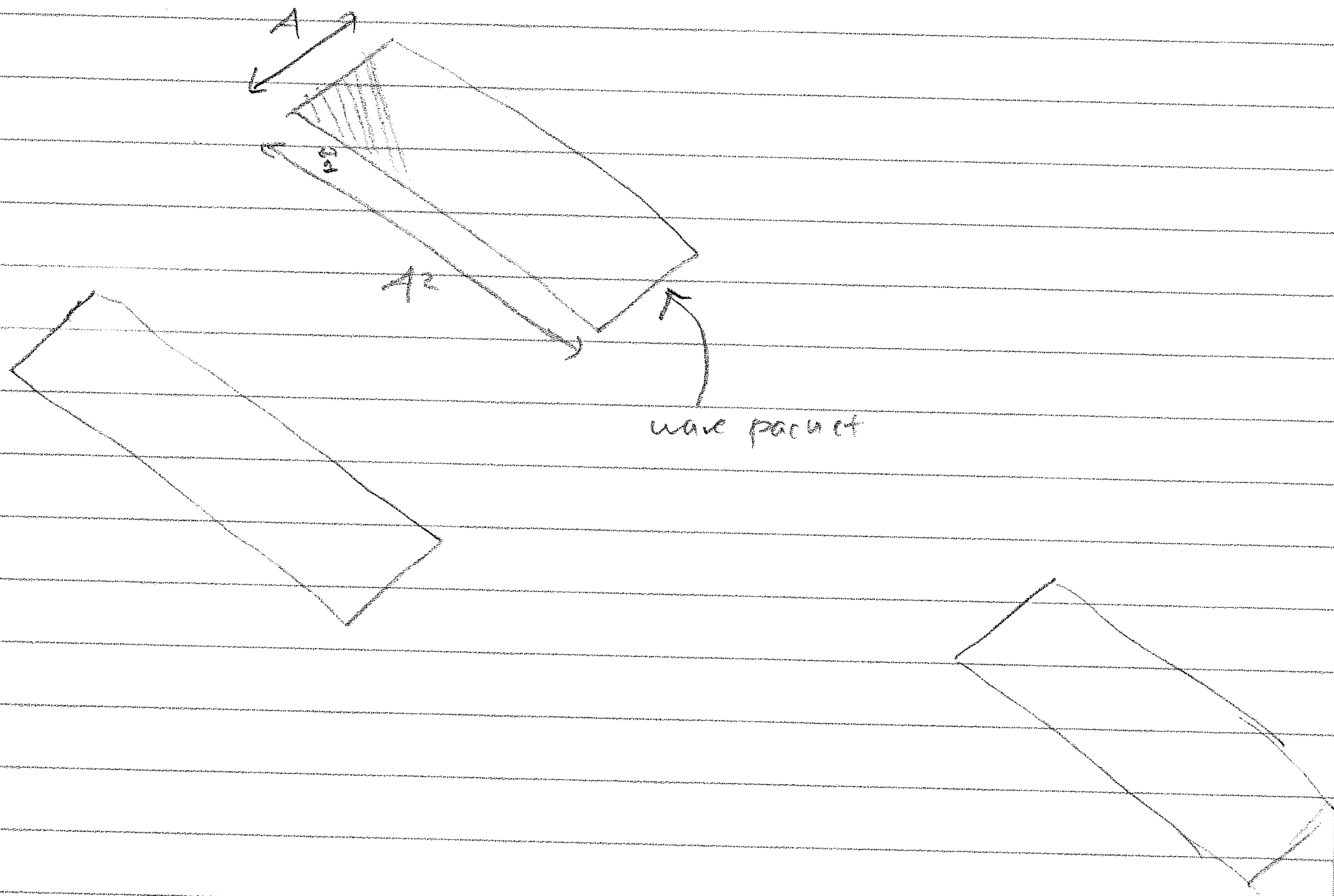
$$\|f\|_{L^\infty(\theta^* + x_0)} \lesssim \|f\|_{L^1(W_{\theta^*+x_0})}$$

(Almost says $|f| \approx$ constant on $\theta^* + x_0$.)

Picture of f_θ



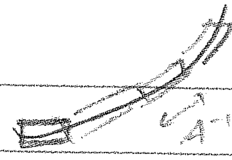
Overhead picture. Draw regions where $|f_\theta|$ is big.



Warning: f_θ oscillates a lot.
frequency is roughly ω_θ, typically $|\omega_\theta| \sim 1$.
draw set where f_θ is pos. real: narrow stripes

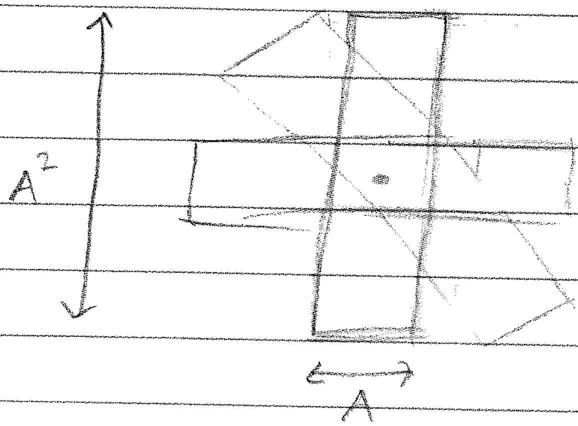
This f_θ is 3 wave packets.

Examples of decoupling



$\#\theta \sim A$

(1) Each f_θ is 1 wave packet, length 1



Arrange at 0, all f_θ pos. real.

- unit ball where $|f| \sim A$.

red $\longleftrightarrow$

$\left[\, |f(x)| \sim \sum_\theta |f_\theta(x)| \quad |f_\theta(x)| \text{ all} \sim \text{equal} \,\right]$

$$\|f_\theta\|_{L^p} \approx (A \cdot A^2)^{\frac{1}{p}} = A^{\frac{3}{p}}.$$

$$\Big(\sum_\theta \|f_\theta\|_{L^p}^2\Big)^{\frac{1}{2}} \approx A^{\frac{1}{2}} \cdot A^{\frac{3}{p}}$$

$$\|f\|_{L^p} \gtrsim A \quad \text{(red dot)}.$$

$$A \lesssim D_p(A) \cdot A^{\frac{1}{2} + \frac{3}{p}}$$

If ~~$p > 6$~~, $D_p(A) \gtrsim A^{\frac{1}{2} - \frac{3}{p}}$

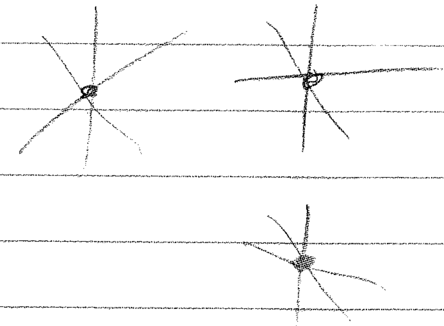
If $p > 6$, > 0.

If $p = 6$, Ex. 1 is sharp (for decoupling ineq.)

(2) $g =$ N disjoint par. copies of ~~Ex.1~~ f (Ex. 1)

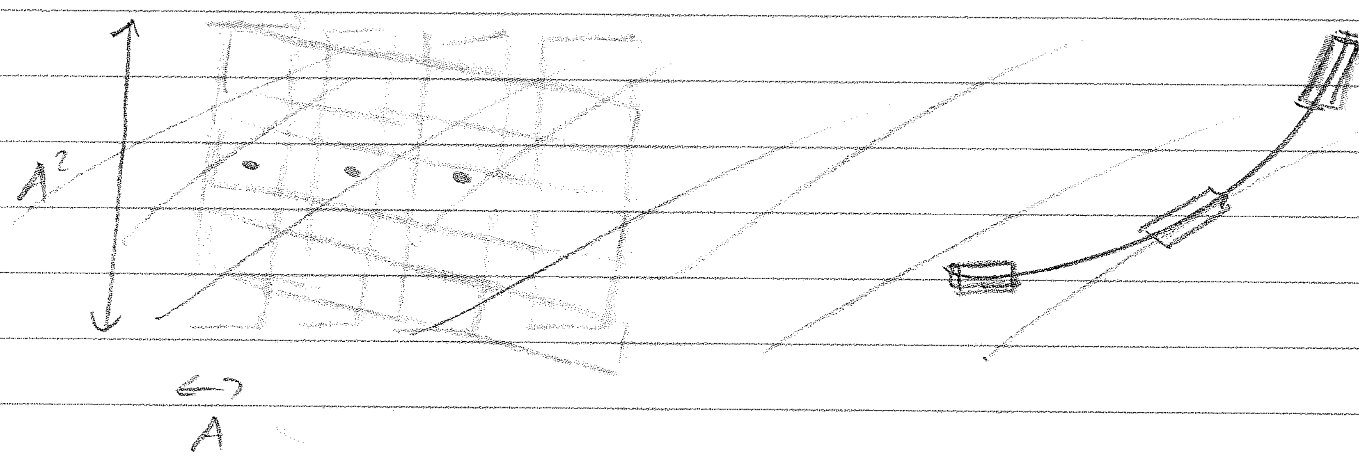
$$\|g_\theta\|_{L^p} = N^{\frac{1}{p}} \cdot \|f_\theta\|_{L^p}$$

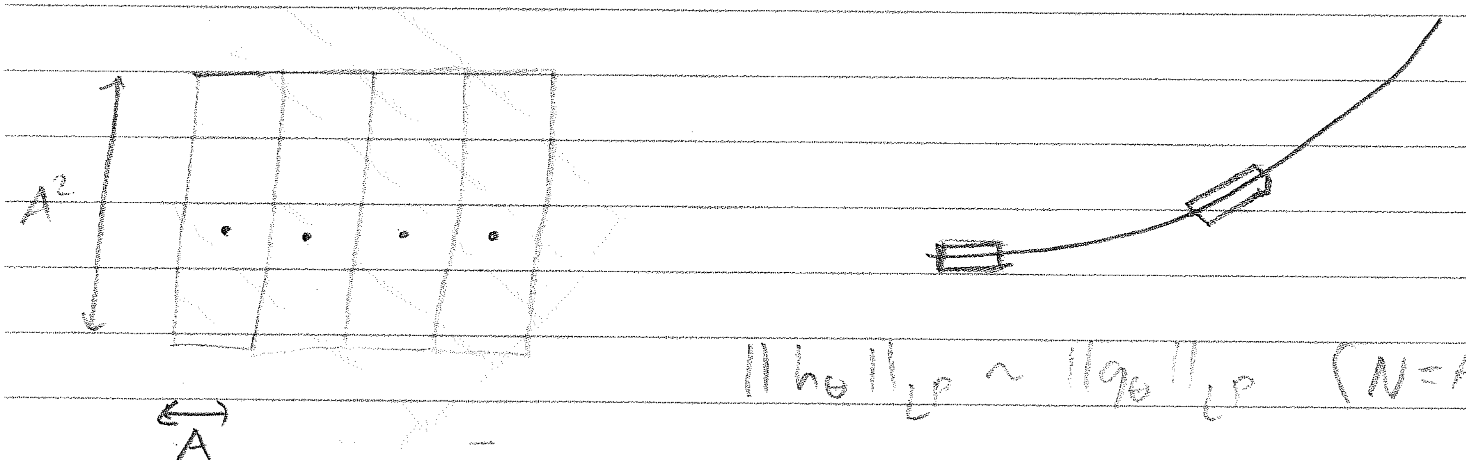
$$\|g\|_{L^p} = N^{\frac{1}{p}} \cdot \|f\|_{L^p}$$



If $p=6$, Ex. 2 is still sharp.

(3) Each h_θ has A wave packets in B_{A^2}.





$$\|h_\theta\|_{L^p} \sim \|g_\theta\|_{L^p} \quad (N=A)$$

$$\|h\|_{L^p} \sim \|g\|_{L^p} \quad (\;''\;)$$

Sharp for $p=6$.

Cor. of Dec. No more red dots!

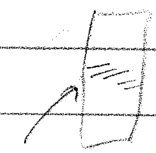
Ingredients of pf.

Orthog.

(Hölder)

Lots of scales!

Geometry of tubes:



area $A \cdot A << A \cdot A^2$

Arranged in an intricate way.

What's really surprising to me: how much mileage get out of looking at many scales, and how many different ways there are to use it.

3 different multi-scale arguments.

(Fantasy: "multi-scale functor" ...)

The statement of decoupling inequality has been carefully crafted to help different scales interact in a nice way.

This Lecture: 1st multi-scale argument

- statement of decoupling crafted to make work.

Lem. If $\Phi : \mathbb{R}^n \circlearrowleft$ is affine isomorphism
then

$$D_p(\tau = \sqcup\theta) = D_p(\Phi(\tau) = \sqcup \Phi(\theta))$$

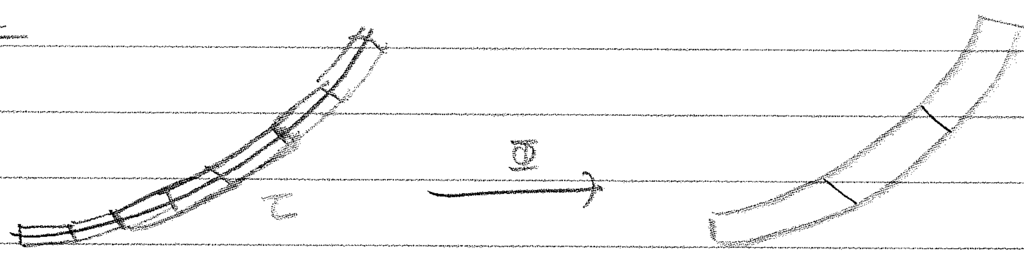
pf. sketch • Φ translation by ω_0

supp $\hat{f} \subset \theta \longleftrightarrow$ supp $\widehat{e^{i\omega_0 x} f} \subset \Phi(\theta)$.

• Φ linear map. $\exists$ linear cov ψ s.t.

supp $\hat{f} \subset \theta \longleftrightarrow$ supp $\widehat{f \circ \psi} \subset \Phi(\theta)$. $\square$

Example.



$A = A_1 \cdot A_2$ $|\theta| = A^{-1}$ $|\tau| = A_1^{-1}$ $\#\theta$ in $\tau = A_2$

$$D_p(\tau = \sqcup\theta) = D_p(A_2)$$

Rmk. If τ is centered at 0,

$$\Phi(\omega_1, \omega_2) = (A_1 \cdot \omega_1, A_1^2 \cdot \omega_2)$$

(If τ is centered elsewhere, a little different.)

Note Choice of dim's θ is $A^{-1} \times A^{-2}$

$\Phi(\theta)$ is $A_1 \cdot A^{-1} \times A_1^2 \cdot A^{-2}$

$= A_2^{-1} \times A_2^{-2}$.

Choice of dim's of θ designed to make this work.

<u>Cor.</u> $D_p(A_1 \cdot A_2) \le D_p(A_1) \cdot D_p(A_2)$

<u>pf.</u> $\|f\|_{L^p} \le D_p(A_1) \cdot \left(\sum_{\substack{\tau \\ |\tau| = A_1^{-1}}} \|f_\tau\|_{L^p}^2 \right)^{\frac{1}{2}}$,

but $\|f_\tau\|_{L^p} \le D_p(A_2) \cdot \left(\sum_{\substack{\theta \subset \tau \\ |\theta| = A^{-1}}} \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}}$.

so $\|f\|_{L^p} \le D_p(A_1) \cdot D_p(A_2) \cdot \left(\sum_\theta \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}}$. $\square$

<u>WRONG Pf. of Main Thm.</u>

Prove $D_p(A) \le A^\varepsilon$ by induction.

$$D_p(A) = D_p(A^{1/2} \cdot A^{1/2}) \le D_p(A^{1/2}) \cdot D_p(A^{1/2})$$

$$\overset{\text{ind.}}{\le} (A^{\frac{1}{2}})^\varepsilon \cdot (A^{\frac{1}{2}})^\varepsilon = A^\varepsilon.$$

What's wrong with pf? No <u>base</u>.

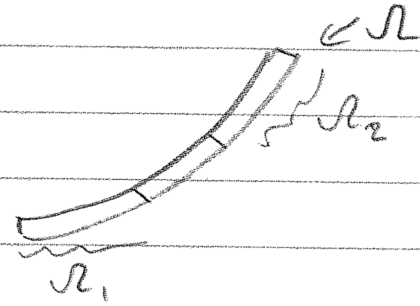
Suspicious: didn't use $2 \le p \le 6$.

<u>Cor.</u> If $\exists A_0$ s.t. $D_p(A_0) \le A_0^\varepsilon$
then $\forall A$, $D_p(A) \lessapprox A^\varepsilon$.

(To prove $D_6(A) \lessapprox A^{\frac{1}{10000}}$ reduces to a finite computation.)

<u>Rmk.</u> For fixed p, A_0, there is an easy algor. (w/ huge running time) to estimate $D_p(A_0)$ to any precision.

Bilinear estimates



$|\Omega_i| \sim 1$

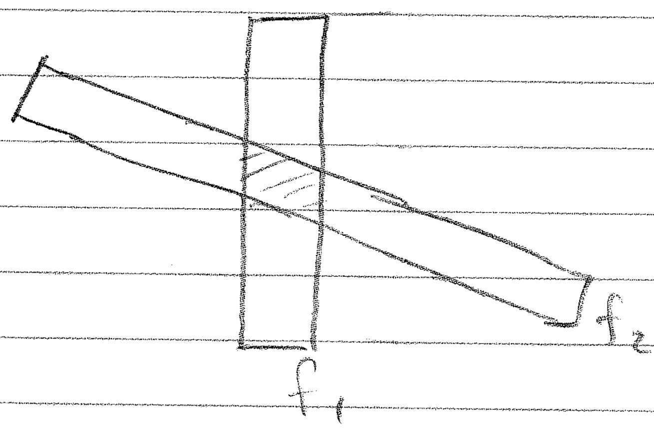
separation ~ 1.

supp $\hat{f}_i \subseteq \Omega_i$.

① $\int |f|^p$ "linear object"

② $\int \prod_{j=1}^{2} |f_j|^{\frac{p}{2}}$ "bilinear object".

Example. f_1, f_2 each single wave packet (from single $\theta_j \subset \Omega_j$)



$$\int \prod_{j=1}^{2} |f_j|^{\frac{p}{2}} << \int |f_1|^p \text{ or } \int |f_2|^p.$$

This phenomenon helps to prove estimates for ②, exploiting our basic fact about geom. of rectangles.

Bilinear decoupling (or Bourgain-Demeter)

$BD_p(A)$ is best constant in:

if $\operatorname{supp} \hat{f_j} \subseteq \Omega_j$, $\quad f_j = \sum_{\substack{\theta \subset \Omega_j \\ |\theta| = A^{-1}}} f_\theta$

then

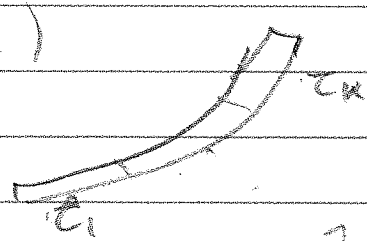
$$\Big\| \prod_{j=1}^{2} |f_j|^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\le BD_p(A) \cdot \prod_{j=1}^{2} \Big[\Big(\sum_{\substack{\theta \subset \Omega_j \\ |\theta| = A^{-1}}} \|f_\theta\|_{L^p}^2 \Big)^{\frac{1}{2}} \Big]^{\frac{1}{2}}$$

Exer. By Hölder $BD_p(A) \le D_p(A)$.

Prop. (BD) $\quad BD_p(A) \lesssim A^{\varepsilon} \cdot D_p(A)$

pf. (broad/narrow trick.)



$$\Omega = \bigsqcup_{K\ \tau\text{'s}} \tau$$

choose K later

$1 \ll K \le A^{\varepsilon}$

Study $\int |f(x)|^p$.

x is **Broad** if $\exists\ \tau_{K_1}, \tau_{K_2}$ non-adjacent s.t.
$|f(x)| \le K^{10} |f_{\tau_K}(x)|$

x is **Narrow** else

$$\int_{\text{Broad}} |f|^p \leq K^{O(1)} \cdot \sum_{\substack{\kappa_1, \kappa_2 \\ \text{nonadj}}} \int_{\mathbb{R}^n} |f_{\tau_{\kappa_1}} \cdot f_{\tau_{\kappa_2}}|^p$$

apply $BD_p(A)$

$$\int_{\text{Narrow}} |f|^p \leq 2 \cdot \sum_{\kappa} \int_{\mathbb{R}^n} |f_{\tau_\kappa}|^p$$

$$\|f\|_{L^p} \leq K^{O(1)} \cdot \sum_{\substack{\kappa_1, \kappa_2 \\ \text{nonadj.}}} \|f_{\tau_{\kappa_1}} \cdot f_{\tau_{\kappa_2}}\|_{L^p}$$

$$+ \left(\sum_{\kappa} \|f_{\tau_\kappa}\|_{L^p}^2 \right)^{\frac{1}{2}} \longleftarrow p \geq 2$$

$$\|f_{\tau_{\kappa_1}} \cdot f_{\tau_{\kappa_2}}\|_{L^p} \leq BD_p(A) \cdot K^{O(1)} \cdot \left(\sum_{|\theta| = A^{-1}} \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}}.$$

$$\|f_{\tau_\kappa}\|_{L^p} \leq D_p\left(\frac{A}{K}\right) \cdot \left(\sum_{\theta \subset \tau_\kappa} \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}}.$$

$\Longrightarrow$

$$D_p(A) \leq K^{O(1)} BD_p(A) + C \cdot D_p\left(\frac{A}{K}\right).$$

By induction $D_p(A) \leq C_0 A^{\varepsilon}$ for all A.
(pick C_0 large so true for small A $\rightarrow$ base, and then apply formula.)