Decoupling Lecture 1

I. Decoupling + analytic number theory

Circle method

$$\{ a_1^n + \dots + a_s^n = b, \quad a_j \in \mathbb{Z}, \quad 0 \le a_j \le A \}.$$

$$r_{n,s,A}(b) := \# \{ \quad \}$$

$$g(y) := \sum_{a=0}^{A} e^{i a^n y} \qquad (1)$$

$$g^s = \sum_{a_1=0}^{A} \dots \sum_{a_s=0}^{A} e^{i(a_1^s + \dots + a_n^s) y}$$

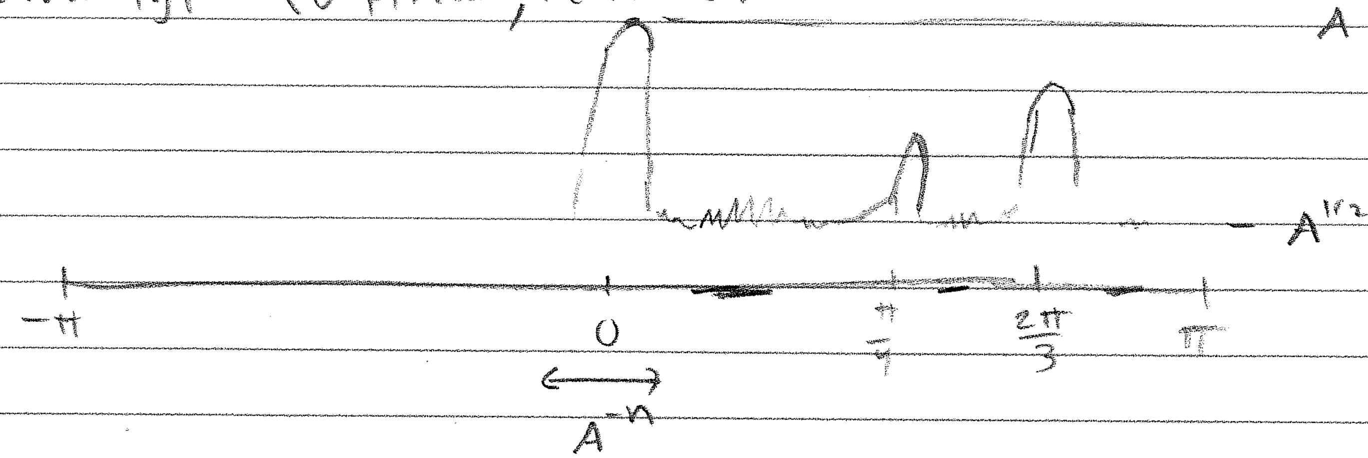
$$= \sum_b r_{n,s,A}(b)\, e^{iby}$$

$$r_{n,s,A}(b) = \widehat{g^s}(b)$$

<u>Plan.</u>

$$\begin{array}{ccc} \text{info } \hat{g} \ (1) & & \text{info } \widehat{(g^s)} \\ \Downarrow & & \Uparrow \\ \text{info } g & \Longrightarrow & \text{info } g^s \end{array}$$

Draw $|g|$ (empirical, heuristic)



$$\int |g|^2 \sim A \qquad \text{(orthog.)}$$

If peak near 0 dominates,

$\Rightarrow r_{k,s,A}(b) \sim$ constant for $|b| \sim A^n$.

Peaks near $\pm\frac{2\pi}{3} \longleftrightarrow$ distrib. of a^n mod 3.

Blue part "noise" appear smaller.

$$D := \{ y \text{ s.t. } \forall\, p, q \in \mathbb{Z}$$

$$\left| y - \frac{p}{q} \right| \gtrsim |q|^{-2-\epsilon} \}.$$

<u>THM.</u> If $y \in D$, $|g(y)| \lesssim A^{1-\sigma_n}$

Weyl 1915	$\sigma_n = \frac{1}{2^{n-1}}$
Vinogradov	$\gtrsim n^{-10}$
$\vdots$	
Wooley	$\approx \frac{1}{2n^2}$
Decoupling	$= \frac{1}{n(n+1)} - \varepsilon$

We can ignore contribution of D if

$$\int_D |g|^s \ll \int_{\text{near}} |g|^s \sim A^{-n+s}$$

$$\downarrow$$

$$\lesssim A^{(1-\varepsilon_n)\cdot s}$$

Happens if s suff. large.
Better $\sigma_n \Rightarrow$ smaller s.

How to use $y \in D$?

Vinogradov symmetry idea

$$f(x_1, \dots, x_n) = \sum_{a=0}^{A} e^{i\left(\frac{a}{A}x_1 + \frac{a^2}{A^2}x_2 + \dots + \frac{a^n}{A^n}x_n\right)}$$

$$g(y) = f(0, \dots, 0, A^n y)$$

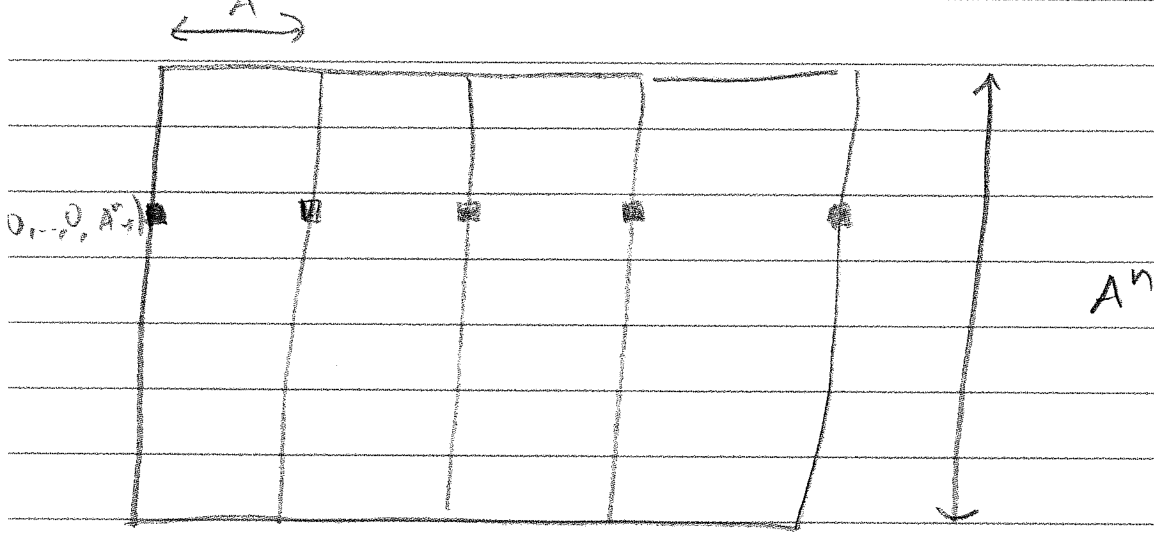
Symmetries of f

f per. in x_1 w/ period $2\pi A$

" x_2 " " $2\pi A^2$

...

$|f| \sim$ constant on balls of rad. 1 (LC.)



Approx. Symmetry.

If $|t| << A$, $t \in \mathbb{Z}$

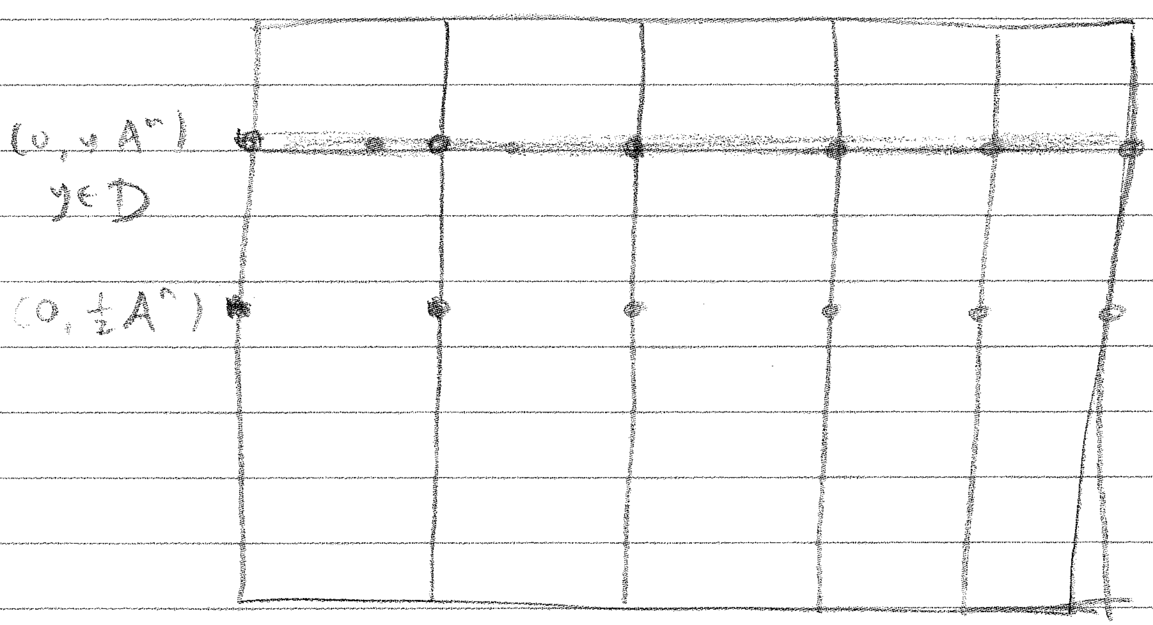
$$f(x) = \sum_{a=0}^{A} e^{i\left(\sum_{j=0}^{n} a^j \frac{x_j}{A^j}\right)} \approx \sum_{a=t}^{A+t} \text{''}$$

$$= \sum_{a=0}^{A} e^{i\left(\sum (a+t)^j \frac{x_j}{A^j}\right)}$$

expand out using binomial

$$= f(\Phi_t(x)) \cdot \text{phase.}$$

Ex. $|f(0, A^2 y)| \approx |f(2tyA, yA^2)|$



$|\text{blue}| = A \cdot |\text{red}|$.

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Estimate $\|f\|_{L^p(B_{A^n})} := \left(\frac{1}{|B_{A^n}|} \cdot \int_{B_{A^n}} |f|^p \right)^{1/p}$

$\|f\|_{L^2(B_{A^n})} \leq A^{1/2}$ orth.

$\|f\|_{L^\infty} \leq A$ (A-idea, bump in red.)

<u>THM</u> (Bourgain - Demeter - G) (many partial results Vinogradov, Wooley, ...)

I) if $p \leq n(n+1)$, $\|f\|_{L^p} \lessapprox A^{1/2+\varepsilon}$

II) if $p \geq n(n+1)$, $\|f\|^p_{L^p(B_{A^n})} \lessapprox A^{\varepsilon} \cdot \int_{\text{red orbit}} |f|^p$

If $y \in D$, $p \in \text{II}$,

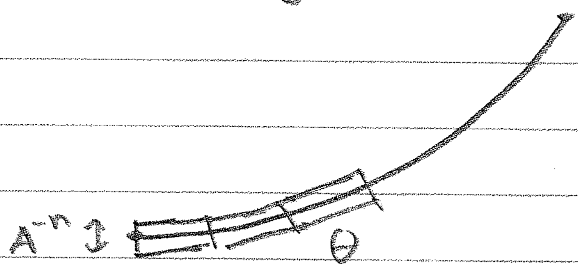
$|\text{blue}| \cdot |g(y)|^p \approx \int_{|\text{blue}|} |f|^p \lessapprox A^{\varepsilon} \int_{\text{red}} |f|^p = |\text{red}| \cdot A^{p+\varepsilon}$.

$|g(y)|^p \leq A^{p-1+\varepsilon}$

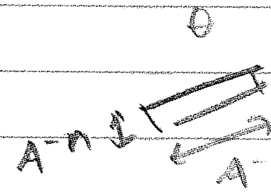
$|g(y)| \leq A^{1-\frac{1}{p}+\varepsilon}$.

Key case $p = n(n+1)$.

Decoupling



Γ = moment curve
$t \longmapsto (t, t^2, \ldots, t^n)$



$N_{A^{-n}}(\Gamma) = \bigsqcup \theta \qquad (\#\theta = A)$

THM. (Decoupling) If supp $\hat{f}_\theta \subseteq \theta$, $f = \sum_\theta f_\theta$
and $2 \le p \le n(n+1)$
then

(D) $$\|f\|_{L^p(B_{A^n})} \lessapprox A^{\epsilon} \cdot \left(\sum_\theta \|f_\theta\|^2_{L^p(W_{B_{A^n}})} \right)^{\frac{1}{2}}$$

$W_{B_{A^n}}$ weight fn. ~ 1 on B_{A^n}, rap. decays.

Rmk. Can put $\mathcal{L}^p$ on both sides.

For Vinogradov f, f_θ = complex expn $e^{i\omega_\theta x}$.
$\|f_\theta\|_{\mathcal{L}^p(W_{B_{A^n}})} \sim 1$.

(D) $\Rightarrow \|f\|_{\mathcal{L}^p} \lessapprox A^{\frac{1}{2}+\epsilon}$. ✓

Focus on $n=2$ (BD 2 yrs. ago)

II. Study building blocks f_θ.

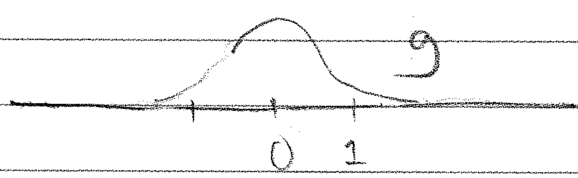
Def. If $\tau \subset \mathbb{R}^n$

$$\mathcal{F}(\tau) := \{ f \mid \text{supp}\, \hat{f} \subset \tau \}.$$

"Fourier of tau".

$\mathcal{F}(B^n(0,1))$

Ex. 1 η bump supp. in $B^n(0,1)$. $g = \check{\eta}$.



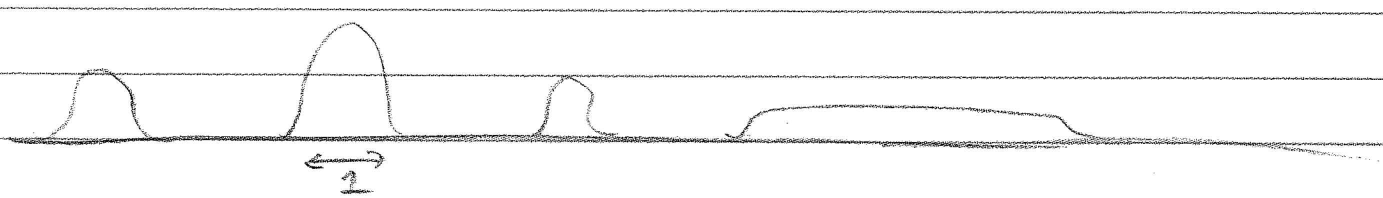
Ex. 2. Translation. $f_{+v}(x) := f(x+v)$

Lemma $f_{+v} \in \mathcal{F}(\tau) \longleftrightarrow f \in \mathcal{F}(\tau)$.

Ex. 3. Linear comb. of Ex. 1 + 2

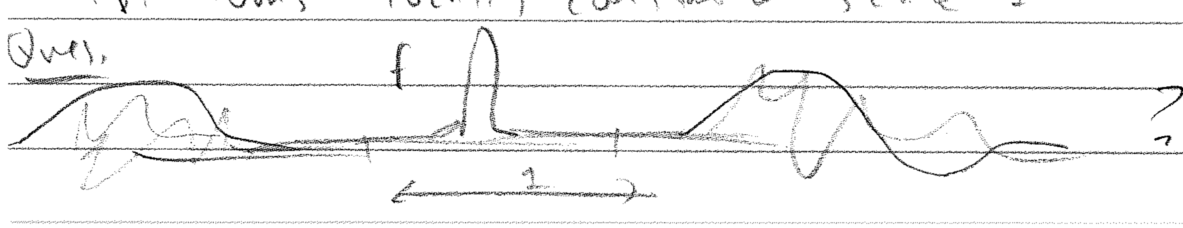
$$f = \sum_v a_v \cdot g_{+v} \qquad a_v \in \mathbb{C}$$

$|f|$



"$|f|$ looks locally constant on scale 1"

Ques.



Let $\psi \in C^\infty_{comp}(\mathbb{R}^n)$, $\psi \equiv 1$ on B_1.

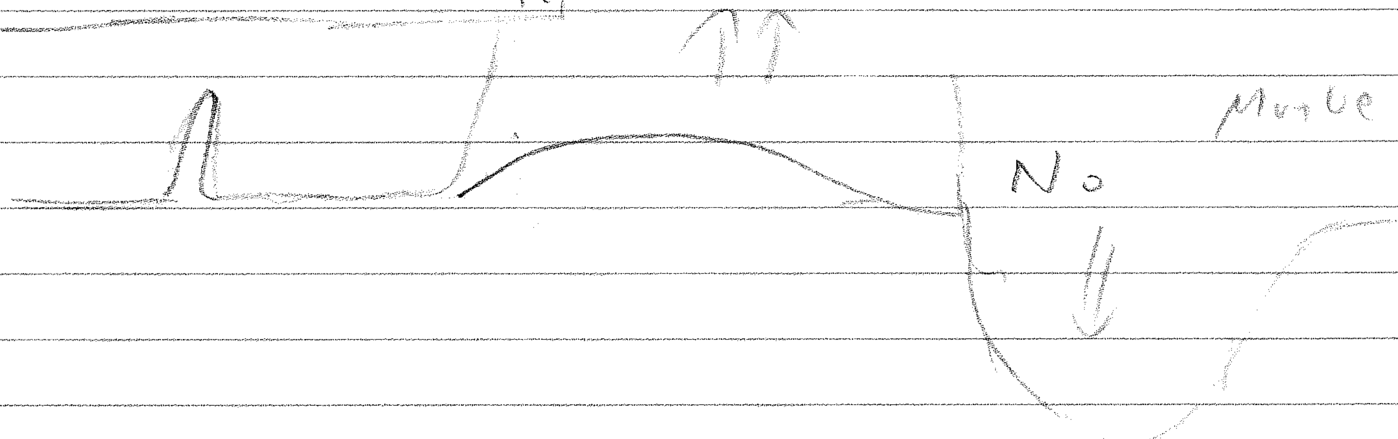
If $f \in \mathcal{F}(B_1)$ then $\hat{f} = \hat{f} \cdot \psi$

$$\Rightarrow \boxed{f = f * \check{\psi}.}$$

$$|\check{\psi}(x)| \lesssim 1 \qquad \forall x$$

rapidly decaying (outside B_1).

$$\boxed{|f| \le |f| * |\check{\psi}|.}$$



<u>Def</u> Weights.

$$W_{B(x_0,1)} = \left(\frac{1}{1+|x-x_0|}\right)^{1000n} \quad \text{(or rap. decaying.)}.$$

$W \sim 1$ on $B(x_0,1)$
decays off

(Locally constant lemma.)

<u>Lem.</u> If $f \in \mathcal{F}(B_1)$, then

$$\|f\|_{L^\infty(B(x_0,1))} \lesssim \|f\|_{L^1(W_{B(x_0,1)})}. \qquad (L^1)$$

<u>pf.</u> $\forall x \in B(x_0,1)$,

$$|f(x)| \le |f| * |\check{\psi}|(x)$$

$$= \int |f(y)|\,|\check{\psi}(x-y)|\,dy \le \text{RHS}.$$

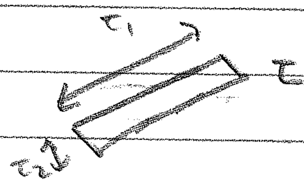
$\mathcal{F}(B^n(w_0,1))$

Lem 2 $f \in \mathcal{F}(B^n(w_0,1)) \longleftrightarrow e^{-iw_0x} f \in \mathcal{F}(B^n(0,1))$

Cor. If $f \in \mathcal{F}(B^n(w_0,1))$, then (LC).

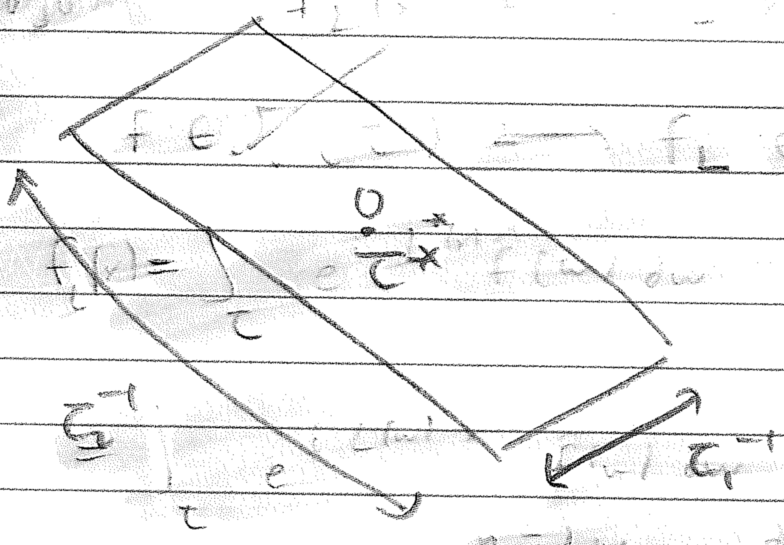
Rmk. cube is similar to a ball.

$\mathcal{F}(\tau)$ τ rect. slab.



Reduces to previous case by C.O.V.

τ^* = "dual" body centered at 0.



$W_{\tau^*}(x) \sim 1$ on τ^*, decays rapidly.

Lem LC3. If $f \in \mathcal{F}(\tau)$ then

$$\|f\|_{L^\infty(\tau^*+x_0)} \lesssim \|f\|_{L^1(W_{\tau^*+x_0})} \quad (LC)$$

Pf. Exer.