Lecture 21: 1 April 2015

21.1 Finishing the Sierpinski Theorem

Recall from our wishful thinking that we wished in part to bound $|\hat{Pf}(n)|$ for $n \in \mathbb{Z}^2$. We also had $\hat{Pf}(n) = \hat{f}(n)$, so the following estimate is useful.

**Proposition 1.** If $f = \chi_{B(R)}$ in $\mathbb{R}^2$, then $|\hat{f}(w)| \lesssim CR^{1/2}|w|^{-3/2}$.

**Proof Sketch.** We have that by rotational invariance,

$$|\hat{f}(w)| = \left| \int_{B(R)} e^{iw \cdot x} dx \right|$$

$$= \left| \int_{B(R)} e^{i|w|x_1} dx_1 dx_2 \right|$$

$$= 2 \left| \int_{-R}^{R} (R^2 - x^2)^{1/2} e^{i|w|x} dx \right|$$

We first integrate by parts with $u = (R^2 - x^2)^{1/2}$ and $dv = e^{iwx}dx$ to obtain

$$|\hat{f}(w)| = \frac{1}{|w|} \left| \int_{-R}^{R} (R^2 - x^2)^{1/2} xe^{iwx} dx \right|.$$ 

If we apply the triangle inequality immediately, we only obtain a bound of about $Rw^{-1}$. Instead, we would like something better. The idea is to break up the domain $[-R,R]$ into two regions, one an interval of the form $[-S,S]$ and the other the set with $S < |x| < R$. On the former region, we can integrate by parts again, and on the outer region we can use the triangle inequality, and then optimize the result by changing $S$. It turns out that taking $S = 1/|w|$ is the right choice, but either way, we get the estimate in the proposition.

**Remark 2.** Also, of course, $|\hat{f}(w)| \leq \pi R^2$ by the triangle inequality directly, and this is a better estimate for small $R$.

Now, we have $N(R) = Pf(0)$, and our wishful thinking would have us hope that this is $\pi R^2 + \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \hat{f}(n)$. But this sum isn’t even absolutely summable given our estimate $|\hat{f}(n)| \lesssim R^{1/2}|n|^{-3/2}$. Instead, we need to do something slightly different. Our trick is to approximate $f$ by smooth functions so that we get better estimates and convergent sums. In order to do this, consider $\eta$ a smooth bump function with support in $B(1)$ and $\int \eta = 1$. Set $f_{R,\epsilon} := \chi_{B(R)} * \eta_\epsilon$, where $\eta_\epsilon(x) = \epsilon^{-2}\eta(x/\epsilon)$ still has integral 1, but the support becomes more and more localized to 0 as $\epsilon \to 0$. For now, we will fix $R$ and simply write $f_\epsilon := f_{R,\epsilon}$, and we will go back after the next proposition. The function $f_\epsilon$ is now smooth and compactly supported, and so we get the convergence properties needed in the sums we are looking at, and in particular, we can prove the following:
Proposition 3. \(|P f_\epsilon(0) - \pi R^2| \leq CR^{1/2}w^{-1/2}|.

Proof. Because \(f_\epsilon\) is smooth, any sum we write in this proof will converge, as the reader can verify. Recall that the Fourier transform commutes with convolution, so we have

\[ Pf_\epsilon(0) = \sum_{n \in \mathbb{Z}^2} \widehat{P f_\epsilon}(n) = \sum_{n \in \mathbb{Z}^2} \widehat{f_\epsilon}(n) \widehat{\eta_\epsilon}(n). \]

The term with \(n = 0\) is just \(\pi R^2\), and so the left hand side of the inequality of our proposition is just

\[ \left| \sum_{n \in \mathbb{Z}^2 - \{0\}} \widehat{f}(n)\widehat{\eta_\epsilon}(n) \right|. \]

Now, since \(\eta\) is compactly supported, we get \(|\widehat{\eta_\epsilon}(n)| \lesssim (1 + |n|\epsilon)^{-10000}\), and so applying the triangle inequality to the sum at hand and using our estimates for \(|\widehat{f}(n)|\) and \(|\widehat{\eta_\epsilon}(n)|\), we find that

\[ |P f_\epsilon(0) - \pi R^2| \lesssim R^{1/2} \sum_{n \neq 0} |n|^{-3/2}(1 + |n|\epsilon)^{-10000} \sim R^{1/2} \sum_{0 < |n| < \epsilon^{-1}} |n|^{-3/2} \sim R^{1/2}w^{-1/2}. \]

Proof of Sierpinski Theorem. Note that \(f_\epsilon \geq 1\) on \(B(R - \epsilon)\), and so we have \(N(R) \leq Pf_{R + \epsilon, \epsilon}(0)\), and so we find

\[ N(R) \leq \pi(R + \epsilon)^2 + CR^{1/2}\epsilon^{-1/2} \]

\[ = \pi R^2 + CR\epsilon + CR^{1/2}\epsilon^{-1/2} \]

\[ \leq \pi R^2 + CR^{2/3} \]

where the last inequality comes from choosing \(\epsilon = R^{-1/3}\) (which is the best possible choice by AM-GM). There is a similar lower bound, and so we find \(|E(R)| \lesssim R^{2/3}\) as desired. \(\square\)