Decoupling course outline

Decoupling theory is a recent development in Fourier analysis with applications in partial differential equations and analytic number theory. It studies the "interference patterns" that occur when we add up functions whose Fourier transforms are supported in different regions. The geometry of the regions in Fourier space influence how much constructive interference can happen in physical space.

There was a recent breakthrough by Bourgain and Demeter, building on previous work by Wolff, Bennett-Carbery-Tao, and others. Before the recent breakthrough, the problems seemed delicate and out of reach. Special cases of the problems could be solved using number theory, and seemed to depend heavily on number theory. For more general cases, it wasn't clear how to bring number theory into play and it sounded hard to capture that kind of fine detail without number theory. The method of proof came as a big surprise to me and many people in the community. The ingredients come entirely from Fourier analysis and geometry. They are all fairly classical and not that difficult. They are combined in an intricate induction argument, and the key point is to take advantage of many different scales.

Here are some of the main applications.

(1) Given a solution of the Schrodinger equation, what can we say about the region where the solution is bigger than some threshold? The Strichartz estimate gives a bound for the volume of the region. We will also study the shape of the region. As a byproduct, decoupling gives good estimates for periodic solutions of the Schrodinger equation.
(2) Given a solution of the wave equation with initial data bounded in some Sobolev norm, what can we say about the Sobolev norm of the solution over a finite time interval?
(3) There is an old trick for writing the number of solutions of some systems of diophantine equations as the L^p norms of some trigonometric sums. Decoupling gives a new tool to estimate such an L^p norm, leading to new (sometimes sharp) bounds for the number of solutions to some diophantine systems.
(4) Vinogradov studied a particular diophantine system, which he used to give greatly improved estimates for the size of exponential sums. Decoupling leads to a sharp estimate for the diophantine systems, giving further small improvements to the exponential sum bounds.

The goal of the class is to study this recent breakthrough and its applications. Along the way, we'll try to give some background on the different problems and fields that are involved.

In this course outline, I'll first describe some of the applications of decoupling, and then go back and explain what decoupling theorems say. After that, I'll give a brief overview of the course.

1. Applications in partial differential equations

The first application that Bourgain and Demeter gave was to prove Strichartz-type estimates for the Schrodinger equation with periodic boundary conditions. Recall that the Schrodinger equation is the PDE

$$\partial_t u = i \triangle u.$$

A basic fact about solutions of the Schrodinger equation is that the L^2 norm is conserved: if $u(x,t)$ solves the Schrodinger equation on $\mathbb{R}^d \times \mathbb{R}$, then

$$\int_{\mathbb{R}^d} |u(x,t)|^2 dx \text{ is constant in } t.$$

While the total L^2 mass is conserved, the mass can move around over time. It's important to understand how much this mass distribution focuses at different times. There is an important inequality called the Strichartz inequality which says that the solution u cannot be large at too many places in space-time:

Theorem 1. *(Strichartz 1970s) If $u(x,t)$ is a solution of the Schrodinger equation on $\mathbb{R}^d \times \mathbb{R}$, then*

$$\|u\|_{L^p(\mathbb{R}^d \times \mathbb{R})} \lesssim \|u(\cdot, 0)\|_{L^2(\mathbb{R}^d)}, \text{ for } p = \frac{2(d+2)}{d}.$$

This estimate plays a crucial role in PDE, for both the linear Schrodinger equation and for non-linear versions.

It's much more delicate to understand how solutions to the Schodinger equation with periodic boundary conditions behave. Let us we define $T^d = \mathbb{R}^d / \mathbb{Z}^d$, the unit cube d-dimensional torus. We consider solutions to the Schrodinger equation on this torus. Bourgain wrote some influential papers on this topic in the early 90s. Informally speaking, the solution of the Schrodinger equation includes waves that travel around the torus in different directions, and the way these different waves add up is hard to estimate. Bourgain was able to prove sharp estimates only in low dimensions $d = 2, 3$, and the proof involved some number theory. It uses unique factorization of integers in order to estimate the number of solutions of some diophantine equations. For higher dimensions, the problem seemed out of reach. Presumably it required some mix of number theory and Fourier analysis and there was no method in the literature that seemed at all promising.

As a corollary of decoupling, Bourgain and Demeter proved essentially sharp estimates for all dimensions.

Theorem 2. *(Periodic Strichartz estimate – Bourgain, Demeter) If $u(x,t)$ is a solution of the Schrodinger equation on $T^d \times \mathbb{R}$, and if the initial data has frequency*

$\sim N$, *then*

$$\|u\|_{L^p(T^d\times[0,1])} \lesssim_\epsilon N^\epsilon \|u(\cdot,0)\|_{L^2(T^d)}, \text{ for } p = \frac{2(d+2)}{d}.$$

A second application has to do with solutions of the wave equation on $\mathbb{R}^d \times \mathbb{R}$. Recall that the initial value problem for the wave equation asks to solve

$$\partial_t^2 u = \triangle u$$

with initial data

$$u(x,0) = f(x) \text{ and } \partial_t u(x,0) = g(x).$$

The problem is the following: if the initial data f, g are bounded in some Sobolev norms, then what can we say about the Sobolev norms of the solution? I was quite surprised when I learned that this fundamantal problem is open. Sogge formulated some conjectures on this topic in the 90s, and they remain open even in dimension $d = 2$. The most recent progress on the problem is based on decoupling.

2. Applications in number theory

In Bourgain's early work on the Strichartz inequality on tori in the 90s, he used a lemma from number theory: a bound on the number of solutions to some diophantine equations. Decoupling implies a sharp Strichartz inequality on tori, and as a corollary, it also gave a new proof for the number theory lemma used in the first proof. From a number theory point of view, this lemma is not that difficult, but it was still quite surprising that one could prove such an estimate without using tools like unique factorization. The decoupling method developed further in work by Bourgain, Demeter, me, and others, and it led to new estimates about the number of solutions to diophantine systems. In particular, it has led to near-sharp estimates for the number of solutions to a system proposed by Vinogradov in the 1930s.

Theorem 3. *(Bourgain-Demeter-Guth, Wooley) Consider the diophantine system*

$$a_1^j + \ldots + a_s^j = b_1^j + \ldots + b_s^j \text{ for all } 1 \le j \le k.$$

Let $J_{s,k}(A)$ be the number of solutions to this system of equations with integers a_i, b_i in the range $[1, A]$. Then

$$J_{s,k}(A) \lesssim_\epsilon A^\epsilon \max(A^s, A^{2s-\frac{k(k+1)}{2}}).$$

The estimate here is sharp up to the factor A^ϵ. This problem originated in Vinogradov's work on exponential sums. Recall the notation $e(x) = e^{2\pi i x}$, and consider the following exponential sum:

$$\sum_{n=1}^{N} e(\alpha n^k).$$

How much cancellation occurs in this sum? In other words, how big is the absolute value of the sum? The answer depends on α. Of course, if α is zero, then each term in the sum is 1, and there is no cancellation. If α is, say, $2/3$, then the terms of the sum are periodic with period 3. Depending on k it might sometimes happen that the sum over one period, $\sum_{n=1}^{3} e(\frac{2}{3}n^k)$, vanishes. But this doesn't happen for all k. When the sum over a period is non-zero, then the exponential sum has size $\sim N$. We expect more cancellation to happen when α is irrational, and in particular when α is hard to approximate by rationals. Recall that α is called diophantine if

$$\left|\alpha - \frac{p}{q}\right| \gtrsim q^{-2} \text{ for all rational numbers } \frac{p}{q}.$$

If α is diophantine, then one expects (almost) square-root cancellation in the exponential sum. This is known to be true for degree $k = 2$, but it is a very hard problem for larger k. Estimates about exponential sums of this type occur in many places in number theory, including estimates for the number of solutions of diophantine equations in the circled method and estimates for the Riemann zeta function and hence estimates about prime numbers. Weyl and van der Corput proved the first non-trivial estimates for theses sums. The proof goes by induction on k and leads to a bad dependence on k.

Theorem 4. *(Weyl, van der Corput) If α is diophantine, then*

$$\left|\sum_{n=1}^{N} e(\alpha n^k)\right| \lesssim N^{1-\gamma(k)}, \text{ for } \gamma(k) = 2^{1-k}.$$

Since the trivial upper bound is N, the improvement in the exponent is $\gamma(k) = 2^{1-k}$. For $k = 2$ this estimate is actually sharp, but for large k the improvement is quite small. Vinogradov in the 1930s introduced a completely different method to estimate exponential sums, the mean value method, which involved estimating the number of solutions to the diophantine system above. Vinogradov gave very good but not quite sharp estimates for the system above, leading to exponential sum estimates with $\gamma(k) \sim \frac{1}{k^2 \log k}$. Over the years since then, there have been a number of incremental improvements in the bounds in Vinogradov's method. Wooley has been the leader in this direction. In the 90s, he proved an exponential sum estimate with $\gamma(k)$ roughly $\frac{2}{k^2}$. Decoupling has given sharp estimates for the diophantine system, and shortly afterwards Wooley proved the same estimates by a different method.

These results give $\gamma(k) = \frac{1}{k(k-1)}$. This seems to be as far as Vinogradov's approach can be pushed.

3. What is decoupling?

Decoupling grew out of the restriction theory for the Fourier transform. Restriction theory asks the following question: suppose that f is a function on $\mathbb{R}^n$ and the Fourier transform of f is supported on some subset $\Omega \subset \mathbb{R}^n$. In terms of the geometry of Ω, what can we conclude about f? The set Ω may be a sphere, a paraboloid, a cone, or a small neighborhood of one of these. If the Fourier transform of f is supported on Ω, then of course we may write

$$f(x) = \int_\Omega e^{i\omega x} \hat{f}(\omega) d\omega.$$

Here is a typical question of restriction theory, raised by Stein in the late 60s. Let P be the truncated paraboloid defined by $\omega_n = \sum_{i=1}^{n-1} \omega_i^2$ and $0 \leq \omega_n \leq 1$. Suppose that $\hat{f}$ is supported in P (as a distribution), and that $\hat{f}$ has the form $g(\omega) d\sigma_P$, where $d\sigma_P$ is the surface area measure on P, and where $\|g\|_{L^\infty(P)} \leq 1$. What can we conclude about $\|f\|_{L^p}$? In dimension $n = 2$, this problem was solved by Fefferman, but in dimensions $n \geq 3$, the problem remains open in spite of a lot of work by many harmonic analysts.

Decoupling theory takes place in this setting. We decompose a function f into pieces with Fourier support in different regions, and then we try to understand the interaction between the pieces. If Ω is a disjoint union of subsets θ, and $\hat{f}$ is supported in Ω, then we can decompose f as

$$f = \sum_\theta f_\theta,$$

where

$$f_\theta(x) = \int_\theta e^{i\omega x} \hat{f}(\omega) d\omega.$$

The problem is to understand how the L^p norm of f relates to the L^p norms of f_θ. The estimates depend on the geometry of Ω and the way that Ω is decomposed into pieces θ.

Here is the first particular case that was proven by Bourgain and Demeter.

Theorem 5. *(Decoupling for the paraboloid, Bourgain-Demeter) Let Ω be the $1/R$ neighborhood of the truncated paraboloid P. Cover Ω with plates θ that are essentially rectanglular solids of dimensions $R^{-1/2} \times ... \times R^{-1/2} \times R^{-1}$. Suppose that $\hat{f}$ is supported in Ω and define f_θ as above. Then*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\epsilon R^\epsilon \left(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{1/2} \quad for\ 2 \le p \le \frac{2(n+1)}{n-1}.$$

An important special case is when f is a trigonometric sum of the form $f(x) = \sum_j a_j e^{i\omega_j x}$. If one frequency ω_j lies in each θ, then the f_θ are just the terms of the trigonometric sum. Strictly speaking, the right-hand side of the inequality in the theorem is infinite in this case, but with a small extra trick this theorem leads to L^p estimates for trigonometric sums over large balls. These estimates immediately give the periodic Strichartz inequality. Similar estimates using the moment curve in place of the paraboloid give the estimates on Vinogradov's diophantine system.

The proof of Theorem 5 came as a big surprise to me and to many people in the community. The conjecture that they proved sounded very difficult – perhaps even harder than the restriction conjecture – and there was no consensus that the conjecture should be true. Because of the connection to periodic Strichartz it was generally believed that number theory would have to be involved in proving such a conjecture, but it was unclear how.

It turned out that the ingredients going into the proof were rather simple - I'll describe them below. These ingredients are combined in an intricate induction argument. The key point is to look at the problem at many scales. The proof actually uses four different ways of combining information from many scales. Because the induction has several layers, it is rather hard to digest, even though it's not very long. We will spend a good amount of time digesting it.

The ingredients that we use at many scales come from Fourier analysis and geometry. On the Fourier analysis we use orthogonality heavily, and also a little bit of convolution and integration by parts. On the geometry side, we use the following type of inequality, due to Loomis and Whitney:

Theorem 6. *(Loomis and Whitney, 1951) Suppose that $X \subset \mathbb{R}^3$ and the coordinate projections of X to each coordinate plane have area at most A. Then the volume of X is at most $A^{3/2}$.*

Informally, this theorem says that if a set X appears to be small when viewed from a number of angles, then the set X must actually be small. The theorem is sharp when X is a cube of side length $A^{1/2}$.

Using these geometric and analytic tools at many different scales and combining the information wisely leads to the decoupling theorems and their applications in PDE and number theory.

4. Class outline

After one or two lectures of introduction, we're going to dive in and start studying the Bourgain-Demeter decoupling theorem for the paraboloid. I think it will take about a month to think through the proof at a leisurely pace. Along the way, we'll introduce background and tools from restriction theory. Next we'll spend some time studying applications of decoupling in PDE and harmonic analysis, including the ones above.

In the next part of the class, we'll switch to number theory topics. We'll give an introduction to how exponential sums figure in number theory, especially in the circle method for estimating the number of solutions to diophantine equations, and we'll explain Vinogradov's approach. At the end of the class we'll discuss decoupling for the moment curve and the resulting bounds for the Vinogradov system and exponential sums.

The class requires a solid background in Fourier analysis (at the level of 18.155-156). I plan to give 4-5 problem sets over the course of the semester to help digest the material. I would also be interested in having participants scribe some of the lectures. We'll talk about that more when the course starts.