We begin with the same basic context as from Lecture 7 for our weak version of decoupling for the truncated paraboloid in $\mathbb{R}^n$. To recall notation:

- $P = \{ (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n : \omega_n = \omega_1^2 + \ldots + \omega_{n-1}^2, |\omega| \leq 1 \}$ the truncated paraboloid;
- $\Omega = N_{1/R}P$ the $R^{-1}$ neighborhood;
- $\Omega = \bigsqcup \theta$ with $\theta$ being $R^{-1/2}$ caps;
- $D_{p,n}(R)$ the decoupling constant.

We are in the middle of proving the following:

**Theorem 0.1** (Bourgain). For $2 \leq p \leq \frac{2n}{n-1}$ and all $\epsilon > 0$, $D_{p,n}(R) \lesssim R^\epsilon$. That is, for any Schwarz $f$ with fourier support in $\Omega$,

$$
\|f\|_{L^p(\mathbb{R}^n)} \lesssim R^\epsilon \left( \sum_{\theta} \|f_{\theta}\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}
$$

1. Reduction to Multilinear Decoupling

We are going to prove Theorem 0.1 using the analogous result in the multilinear context. Suppose $f_j$ have Fourier support in $1/R$ neighborhoods $\Omega_j$ of surfaces $\Sigma_j$ with normals close to the $x_j$-axis and divided into $R^{-1/2}$-caps $\theta$, and set $MD_{p,n}(R)$ the best constant for the inequality:

$$
\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p(\mathbb{R}^n)} \leq MD_{p,n}(R) \prod_{j=1}^n \left( \sum_{\theta \subset \Sigma_j} \|f_{j,\theta}\|_{L^p(\mathbb{R}^n)} \right)^{1/n}
$$

We have shown, essentially as a corollary of multilinear restriction, the following:

**Proposition 1.1.** For $2 \leq p \leq \frac{2n}{n-1}$ and all $\epsilon > 0$, $MD_{p,n}(R) \lesssim R^\epsilon$.
Unlike the Kakeya or restriction problems, in the decoupling context we are able to turn multilinear decoupling results into standard decoupling results via reduction of dimension arguments. The following is the main result allowing induction on scale and dimension

**Lemma 1.2 (Main Lemma).** For \(2 \leq p \leq \frac{2n}{n-1}\),

\[
D_{p,n}(R) \lesssim K^{O(1)} MD_{p,n}(R) + D_{p,n-1}(K^2)D_{p,n}(R/K^2)
\]

The right-hand side has two terms. The first will be referred to as the **broad** term, and comes from spatial regions where many wave packets from transverse directions contribute allowing use to use multilinear restriction. The latter is the **narrow** term and comes from regions of space where significant contributions come from a lower dimensional space of Fourier values.

Note: the allowable range of \(p\) increases as \(n\) decreases, so we are able to iterate/use induction. Most of the remaining time will be devoted to proving the main lemma. If we use a well chosen \(K\), either \(K \sim \log R\) or \(K = K(\epsilon)\) large, iteration of the main lemma proves Theorem 0.1.

### 2. Broad and Narrow

To prove the main lemma we write \(\Omega = \bigcup \tau\) where \(\tau\) are \(K^{-1}\)-caps. We have \(f = \sum f_{\tau}\). The \(|f_{\tau}|\) are roughly constant on \(K \times K^2\) tubes.

[ Picture 1: For different caps \(\tau_1, \tau_2, \tau_3\ldots\) we have a picture of the tubes in \(\mathbb{R}^n\) where the \(f_{\tau_i}\) are large. In some balls of radius \(K^2\) many \(f_{\tau_i}\) are large. These are broad balls. In others only a small number are, these are narrow.]

We tile \(\mathbb{R}^n\) with balls of radius \(K^2\), \(B = B(x_0, K^2)\). For a ball define the **significant** set

\[
S(B) := \left\{ \tau : \|f_{\tau}\|_{L^p(B)} \geq \frac{1}{100(\#\tau)}\|f\|_{L^p(B)} \right\}
\]

We observe that \(\sum_{\tau \in S(B)} f_{\tau}\) has \(L^p\) norm comparable to \(f\). Now remember that we can apply multilinear restriction to any collection of surfaces with normals that are not too close to being linearly independent.

We say a ball \(B\) is **narrow** if there is a hyperplane \(\Pi^*\) such that for all \(\tau \in S(B)\)

\[
\angle(\text{nor}(\tau), \Pi^*) \lesssim (nK)^{-1}
\]

Otherwise, we say the ball is **broad**, which we note means that there are caps \(\tau_1, \ldots \tau_n \in S(B)\) such that they satisfy the conditions on \(\Sigma_j\) of
multilinear restriction after a linear change of variables with bounded determinant. We shall refer to the union of all broad balls as \textit{Broad} and the union of all narrow balls as \textit{Narrow}.

We now re-write the main lemma as two separate estimates:

**Broad Estimate:**

\[
\|f\|_{L^p(\text{Broad})} \lesssim K^{O(1)} MD_{p,n}(R) \left( \sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}}
\]

**Narrow Estimate:**

\[
\|f\|_{L^p(\text{Narrow})} \lesssim D_{p,n-1}(K^2) \left( \sum_\tau \|f_\tau\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}}
\]

Together, these two estimates imply the main lemma once we observe that by decoupling at scale \(R/K^2\),

\[
\left( \sum_\tau \|f_\tau\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}} \leq D_{p,n}(R/K^2) \left( \sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}}
\]

### 3. Narrow Estimate

We did a large piece of the argument for the narrow estimate in the previous lecture. We proved the following lemma:

**Lemma 3.1.** Suppose \(B\) is narrow, \(\Pi^* = \Pi^*(B)\) the hyperplane such that \(\text{nor}(\tau)\) is close to \(\Pi^*\) for all \(\tau \in S(B)\). Set \(f_B = \sum_{\tau \in S(B)} f_\tau\). For any \(\Pi\) parallel to \(\Pi^*\);

\[
\|f\|_{L^p(B \cap \Pi)} \lesssim D_{p,n-1}(K^2) \left( \sum_{\tau \in S(B)} \|f_\tau\|^2_{L^p(\omega_B \cap \Pi)} \right)^{1/2}
\]

To briefly sketch the ideas, we selected well adapted coordinates \((y_1, \ldots, y_n) = (y', y_n)\) such that \(\Pi_t = \{ y_n = t \}\) are the hyperplanes \(\Pi\) in the lemma. If \(g_B(y') = f_B(y', t)\), \(g_\tau(y') = f_\tau(y', t)\) then the support of \(\hat{g}_\tau\) lies in a paraboloid in \(n-1\) dimension by induction.

[ Picture 2: Drawing of a paraboloid in lower dim'l space where the Fourier support of the \(g\)'s live. Caps are of size \(K^{-1}\) ]

Lemma 3.1 is a decoupling result for a hyperplane slice of a ball. Morally, we immediately get decoupling using the parallel decoupling
argument. We shall do the Minkowski’s inequality argument by hand to recover a local decoupling result on all of $B$ from the estimate on slices, and then quote parallel decoupling to prove the narrow estimate.

$$\|f\|_{L^p(B)}^p \sim \|f_B\|_{L^p(B)}^p$$

(Fubini) $= \int \|f\|_{L^p(B \cap \Pi_t)}^p dt$

(Decoupling) $\lesssim (D_{p,n-1}(K^2))^p \int \left( \sum_{\tau \in s(B)} \|f_{\tau}\|_{L^p(B \cap \Pi_t)}^2 \right)^{p/2} dt$

$\lesssim D^p \sum_{\tau \in S(B)} \int \|f_{\tau}\|_{L^p(B \cap \Pi_t)}^2 \left( \sum_{\tau \in s(B)} \|f_{\tau}\|_{L^p(B \cap \Pi_t)}^2 \right)^{p/2} dt$

(Hölder) $\lesssim D^p \sum_{\tau \in S(B)} \left( \int \|f_{\tau}\|_{L^p(B \cap \Pi_t)}^p dt \right)^{\frac{2}{p}} A^{\frac{p-2}{p}}$

Where here $A = \int \left( \sum_{\tau \in s(B)} \|f_{\tau}\|_{L^p(B \cap \Pi_t)}^2 \right)^{p/2} dt$. Finally, by dividing out or iterating and taking $p$-th roots we get the desired local bound:

$$\|f\|_{L^p(B)} \leq D_{p,n-1}(K^2) \left( \sum_{\tau} \|f_{\tau}\|_{L^p(\omega_B)}^2 \right)^{\frac{1}{2}}$$

Invoking parallel decoupling then gives the narrow estimate.

4. Broad Estimate

There is a subtlety here, so we begin by writing down an incorrect attempt at an argument. We write $(\tau_1, ..., \tau_n) = \vec{\tau} \in T$ if the set of $\tau_i$ are transverse.

$$\int_{\text{Broad}} |f|^p = \sum_{B \subset \text{Broad}} \int_B |f|^p$$

$$\leq \sum_{B \subset \text{Broad}} \sum_{\vec{\tau} \in T} K^{O(1)} \prod_{j=1}^n \left( \int_B |f_{\tau_j}|^p \right)^{\frac{1}{n}}$$

$$\lesssim K^{O(1)} \sum_{\vec{\tau} \in T} \int_{\mathbb{R}^n} \prod_{j=1}^n |f_{\tau_j}|^{p/n} \lesssim ...$$
Here the inequality marked with a "?" is something we would like to say is true, as once we get to the last line we may invoke multilinear decoupling to finish the estimate.

We stop for a moment to consider whether the passage from the second to last line to the last line is valid. Comments made during class were:

- It looks like Hölder in the wrong direction;
- We are bounding a product of integrals by the integral of the products, not usually valid but maybe something about the Fourier support conditions/locally constant properties allow us to use it in this case?

The problem is that $B = B(K^2)$ has too large a radius to invoke the locally constant property, so in fact the functions in question may be large on disjoint sets.

[Picture 3: Two possible scenarios are drawn for how the supports of $f_{\tau_i}$ may interact in broad ball $B$. In 3(a) the transverse tubes on which the functions are large avoid each other and the inequality fails. In 3(b) all of the tubes intersect at the center]

How do we fix this? We are saved by the fact that we only want a bound with some polynomial dependence on $K$. We use random translations of the $f_{\tau_i}$ and with probability $K^{-O(1)}$ the good picture will result. Let $v_1, \ldots, v_n$ be drawn independently uniformly at random from $B(K^2)$. For an arbitrary function $g$ we write $g_v(x) = g(x - v)$.

**Lemma 4.1.** For $\tau_1, \ldots, \tau_n \in S(B)$ transverse:

$$
\|f\|_{L^p(B)}^p \lesssim K^{O(1)} \mathbb{E}_v \left\| \prod_j |f_{\tau_j,v_j}|^{1/n} \right\|_{L^p(B)}^p
$$

We don’t prove this lemma in detail. The main point is that with probability $\sim K^{-O(1)}$, $|f_{\tau_j,v_j}| \sim \|f_{\tau_j}\|_{L^\infty(B)}$ on $B(x_0, K)$ (this being the scale at which we can invoke the locally constant property), and the lemma is immediate.

We note some nice properties of translation operators, the first two of which come from the fact that it is just a phase multiplier in frequency space.

(A) They preserve Fourier support.
(B) They commute with projections $f \mapsto f_\theta$.
(C) They preserve $L^p$ norms.

Using these properties and Lemma 4.1 we then can successfully finish the argument:
\[
\int_{\text{Broad}} |f|^p = \sum_{B \subset \text{Broad}} \int_B |f|^p
\]
\[
\lesssim K^{O(1)} \mathbb{E}_\tau \sum_B \left\| \prod_{j=1}^n |f_{\tau_j, v_j}|^{\frac{1}{n}} \right\|_{L^p(B)}^p
\]
\[
= K^{O(1)} \sum_\tau \mathbb{E}_v \left\| \prod_{j=1}^n |f_{\tau_j, v_j}|^{\frac{1}{n}} \right\|_{L^p(\mathbb{R}^n)}^p
\]
\[
\leq MD_{p,n} K^{O(1)} \sum_\tau \mathbb{E}_v \left( \prod_{j=1}^n \left( \sum_{\theta \in \tau_j} \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2n}} \right)^{\frac{1}{2}}^p
\]

And now some fairly trivial moves: we enlarge the indexing set for \( \Omega \), which removes dependence on \( j \) and probability and we observe that the indexing set for \( \bar{\tau} \) is size some polynomial in \( K \).

\[
\leq MD_{p,n} K^{O(1)} \sum_\tau \mathbb{E}_v \left( \prod_{j=1}^n \left( \sum_{\theta \in \Omega} \|f_\theta\|_{L^p}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2n}}
\]

This concludes the proof of the broad estimate, and thus Theorem 0.1.

5. Final thoughts

Having done a big piece of math some questions came up.

- What does the broad versus narrow decomposition tell us about the Kakeya problem? See the third problemset.
- What is the base case in dimension? We proved the \( n = 2 \) restriction problem using the broad vs. narrow decomposition and so we would directly deduce \( n = 2 \) decoupling.
- What is the actual polynomial in $K$? You can read it out, and optimize the $R^e$ statement, but it doesn’t give anything particularly clean/enlightening.