1. Geometric input

1.1. A Warm up problem.

**Question 1.1.** Suppose $U \subset \mathbb{R}^3$ is a bounded region and

\[
\text{Area}(\text{Proj}_{xy}\text{-plan}(U)) = A,
\text{Area}(\text{Proj}_{xy}\text{-plan}(U)) = B,
\text{Area}(\text{Proj}_{xy}\text{-plan}(U)) = C.
\]

Then how big is the volume of $U$?

To deal with the problem, define

\[U_z = \{(x, y) \in \mathbb{R}^2 | (x, y, z) \in U\},\]

then

\[|U| = \int_\mathbb{R} |U_z|dz,\]

and for any $z \in \mathbb{R}$, $|U_z| \leq A$.

Also let

\[X_z = \{x \in \mathbb{R} | (x, y) \in U_z\}, \quad Y_z = \{y \in \mathbb{R} | (x, y) \in U_z\}.\]

See figure [1] Then we have for any $z \in \mathbb{R}$

\[|U_z| \leq |X_z| \cdot |Y_z|,\]

and

\[\int_\mathbb{R} |X_z|dz \leq B, \quad \int_\mathbb{R} |Y_z|dz \leq C.\]
So altogether by Cauchy-Schwartz

\[ U = \int_{\mathbb{R}} |U_z| dz \]
\[ \leq \int_{\mathbb{R}} A^{\frac{1}{2}} (|X_z| \cdot |Y_z|)^{\frac{1}{2}} dz \]
\[ = A^{\frac{1}{2}} \cdot \int_{\mathbb{R}} |X_z|^{\frac{1}{2}} |Y_z|^{\frac{1}{2}} dz \]
\[ \leq A^{\frac{1}{2}} \cdot \left( \int_{\mathbb{R}} |X_z| dz \right)^{\frac{1}{2}} \cdot \left( \int_{\mathbb{R}} |Y_z| dz \right)^{\frac{1}{2}} \]
\[ \leq (ABC)^{\frac{1}{2}}. \]

Actually this bound is tight: for a hyperrectangle in \( \mathbb{R}^3 \) whose edges are parallel to axes, one can check easily that the equality holds.

This argument also gives us a simple prove of the isoperimetric inequality in dimension 3.

**Corollary 1.2.** Suppose \( U \) is a bounded region in \( \mathbb{R}^3 \), then

\[ |U| \leq |\partial U|^{\frac{3}{2}}. \]

**Proof.** We have

\[ \text{Area}(\text{Proj}(U)) \leq |\partial U| \]

and then apply the above result. \( \square \)

**Theorem 1.3** (Loomis-Whitney, 1949). Suppose we have for \( j = 1, 2, \ldots, n, \)

\[ f_j : \mathbb{R}^{n-1} \to \mathbb{R}^+ \]
a positive valued function and

\[ \pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1} \]

the projection by forgetting the \( j \)-th coordinate, then

\[ \int_{\mathbb{R}^n} \prod_{j=1}^{n} (f_j \circ \pi_j) \frac{1}{n-1} \leq \prod_{j=1}^{n} \left( \int_{\mathbb{R}} f_j \right)^{\frac{1}{n-1}}. \]

Example 1.4. Suppose \( U \subset \mathbb{R}^n \) is a bounded region and let

\[ f_j = \chi_{\pi_j(U)} \]

be the characteristic function of \( \pi_j(U) \subset \mathbb{R}^{n-1} \). Then apply the Loomis-Whitney's theorem we get

\[ |U| \leq LHS \leq \prod_{j=1}^{n} \left( |\pi_j(U)| \right)^{\frac{1}{n-1}}. \]

1.2. Tubes along different directions

Setup 0. Let \( l_{j,a} \subset \mathbb{R}^n \) be lines that are parallel to \( x_i \)-axis for \( 1 \leq j \leq n \) and \( 1 \leq a \leq N_j \). Let \( T_{i,a} \) to be the characteristic function of the 1-neighborhood of \( l_{j,a} \).

With the above setups we have the following estimate about how they overlap.

Corollary 1.5.

\[ \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{l,a} \right)^{\frac{1}{n-1}} \leq \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}, \]

where the constant depends on the dimension \( n \).

Example 1.6. Suppose \( Q_s \) is a cube of size \( s \) and for each direction it is covered by disjoint tubes.

Then for each \( j \), \( N_j \sim s^{n-1} \) and

\[ \sum_{a=1}^{N_j} T_{l,a} = 1. \]

Hence

\[ LHS \sim \int_{Q_s} 1 = s^n, \]

and

\[ RHS = \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} = s^n. \]
Example 1.7. Suppose for any $j$, $l_{j,a}$ are the same for all $a$, and they all pass the origin. Then

$$LHS \sim \int_{B_1} \prod_{j=1}^n N_j^{\frac{1}{n-1}} \sim \prod_{j=1}^n N_j^{\frac{1}{n-1}} = LHS.$$ 

Proof of corollary 1.5. Let $\pi_j$ be the projection as in theorem 1.3 and

$$f_j = \sum_{a=1}^{N_j} \chi_{D_{j,a}} ,$$

where $D_{j,a} \subset \mathbb{R}^{n-1}$ is the disk of radius 1, centered at the point $l_{j,a} \cap \mathbb{R}^{n-1}$. Then we have

$$T_{l,a} = f_j \circ \pi_j.$$ 

Then

$$LHS \text{ of Cor} = \int_{\mathbb{R}^n} \prod_{j=1}^n (f_j \circ \pi_j)^{\frac{1}{n-1}}$$

$$\leq \prod_{j=1}^n \left( \int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n-1}}$$

$$= \prod_{j=1}^n \left( \int_{\mathbb{R}^{n-1}} \sum_{a=1}^{N_j} \chi_{D_{j,a}} \right)^{\frac{1}{n-1}}$$

$$\leq \prod_{j=1}^n N_j^{\frac{1}{n-1}}$$

□

Setup 1. Let $l_{j,a}$ be lines in $\mathbb{R}^n$ with angle between $l_{j,a}$ and $x_j$-axis no larger than $\frac{1}{100n}$. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $l_{j,a}$.

Question. Does inequality (1) also holds for setup 1? If $n = 2$ the answer is yes. In this case the two tubes intersect in a region very close to a unit cube, having area close to 1. See figure 2 (a). Then

$$\int_{\mathbb{R}^2} \left( \sum_{a=1}^{N_1} T_{1,a} \right) \left( \sum_{b=1}^{N_2} T_{2,b} \right) = \sum_{a=1}^{N_1} \sum_{b=1}^{N_2} \int_{\mathbb{R}^2} T_{1,a} \cdot T_{2,b}$$

$$\lesssim N_1 N_2.$$
1.3. Tilting vs bending

Setup bend. Let $\gamma_{j,a}$ be curves in $\mathbb{R}^n$ such that for any point $x \in \gamma_{l,a}$, the angle between the tangent line at $x$ and $x_j$-axis is no larger than $\delta$. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $\gamma_{j,a}$.

The inequality (1) also holds for the case $n = 2$. The proof is similar to the $n = 2$ case in setup 1. See figure 2 (b). However, there is a scary counterexample in dimension 3 (unpublished) by Csormyei that for large enough $N$, there exists $\varepsilon$ depending on $\delta$ and curves $\gamma_{j,a}$ for $1 \leq j \leq 3, 1 \leq a \leq N$ such that

$$\int_{\mathbb{R}^3} \prod_{j=1}^{3} \left( \sum_{a=1}^{N} T_{j,a} \right)^{\frac{1}{2}} \geq N^{\varepsilon + \frac{3}{2}}.$$ 

The tilting case is related to the multi-linear Kakeya problem. A recent result is the following.

**Theorem 1.8** (Bennett-Carbery-Tao, 2005). For setup 1 (the tilting case), suppose $Q_s$ is the cube of size $s$, then for any $\varepsilon > 0$,

$$\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim s^{\varepsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}. \quad (2)$$

Here the constant depends on $n$ and $\varepsilon$.

**Lemma 1.9** (Main lemma). For any $\varepsilon > 0$, there exists $\delta$, such that if all angles between $l_{j,a}$ and $x_j$-axis are no larger than $\delta$, then inequality (2) holds.

**proof of the theorem by lemma 1.9** Suppose $S_j \subset S^{n-1}$ be the region on the unit sphere in $\mathbb{R}^n$, consisting of points such that all lines passing through origin and the point has angle with $x_j$-axis no larger than $\frac{1}{100n}$. 

*Figure 2.* Tilting and bending in dimension 2.
Let
\[ S_j = \bigcup_b S_{j,b}, \]
where each \( S_{j,b} \) has diameter no larger than \( \frac{\delta}{10} \). See figure 3. Also let \( e_j \) be the vector such that only the \( j \)-th coordinate is 1 and all the other coordinates are 0, so \( e_j \) is actually the intersection point of \( x_j \)-axis with \( S^{n-1} \) and also the center of \( S_j \).

![Figure 3. Subdivide \( S_j \) into small pieces.](image)

Define
\[ g_j = \sum_{a=1}^{N_j} T_{j,a}, \quad g_{j,b} = \sum_{l_j,a \cap S^{n-1} \in S_{j,b}} T_{j,a}, \]
then
\[ g_j = \sum_b g_{j,b}. \]

Now
\[ LHS \ of \ theorem = \int_{Q_s} \prod_j g_j^{\frac{1}{n-1}} \]
\[ = \int_{Q_s} \prod_j (\sum_b g_{j,b})^{\frac{1}{n-1}} \]
\[ \leq \sum_{(b_1, \ldots, b_n)} \int_{Q_s} \prod_{j=1}^n g_{j,b_j}^{\frac{1}{n-1}}. \]

First note that the number of choice of \((b_1, \ldots, b_n)\) depends on both \(n\) and \(\delta\). Second, if each \( S_{j,b_j} \) is centered at the point \( e_j \), then we can
apply lemma 1.9 directly. If some $S_{j,b_j}$ is not centered at $e_j$, we can apply a linear change of variables to move the center of $S_{j,b_j}$ to $e_j$. The determinant of the coordinate change is controlled by $\delta$ and hence by $\varepsilon$. Then we have for any $(b_1, ..., b_n)$,
\[
\int_{Q_s} \prod_{j=1}^{n} g_{j,b_j}^{\frac{1}{n}} \lesssim s^\varepsilon \prod_{j=1}^{n} N_{j}^{\frac{1}{n-1}}.
\]
Thus altogether we conclude that
\[
LHS \text{ of theorem } \leq \sum_{(b_1, ..., b_n)} \int_{Q_s} \prod_{j=1}^{n} g_{j,b_j}^{\frac{1}{n}} \lesssim s^\varepsilon \prod_{j=1}^{n} N_{j}^{\frac{1}{n-1}}.
\]

Now let us try to prove the main lemma. First we should look on a cube of size $\delta^{-1}$.

**Lemma 1.10.** If $s \leq \delta^{-1}$, then
\[
\int_{Q_s} \prod_{j=1}^{n} (\sum_{a=1}^{N_j} T_{j,a})^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}},
\]
where the constant depends on $n$, and
\[
N_j(Q_s) = \#\{a | \text{supp}(T_{j,a}) \cap Q_s \neq \phi\}.
\]

**Proof.** Since $s \leq \delta^{-1}$ we have
\[
\text{supp}(T_{j,a}) \cap Q_s \subset \tilde{T}_{j,a},
\]
where $\tilde{T}_{j,a}$ is a neighborhood of lines parallel to $x_j$-axis of radius $\sim 1$. See figure 4. We slightly abuse the notation to use $\tilde{T}_{j,a}$ also denote the characteristic function of the region. Then by theorem 1.3, we have
\[
LHS \leq \int_{Q_s} \prod_{j=1}^{n} (\sum_{a=1}^{N_j} \tilde{T}_{j,a})^{\frac{1}{n-1}} \lesssim \text{RHS}.
\]

Now we want to deal with a cube of larger size. The basic idea is to divide the large cube into small ones of size we have already dealt with. To do this, define $T_{j,a,w}$ to be the characteristic function of $w$-neighborhood of the line $l_{j,a}$ so $T_{j,a,1} = T_{j,a}$.

**Lemma 1.11.** Suppose $Q_s$ is a cube of size
\[
\frac{1}{20n} \delta^{-1} \leq s \leq \frac{1}{10n} \delta^{-1},
\]
then we have
\[ \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq \delta^n \cdot \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}}, \]
where the constant depends on \( n \).

**Proof.** Give lemma 1.10, we need only to check
\[ \text{RHS} \geq \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}}. \]

Observe that if \( s \leq \frac{1}{10} \delta^{-1} \) and \( \text{supp}(T_{j,a}) \cap Q_s \neq \phi \), then \( T_{j,a,\delta^{-1}} \equiv 1 \) on \( Q_s \). See figure 5.

**Figure 4.** \( T_{j,a} \) is contained in a slightly larger tubes that are parallel to axis.

**Figure 5.** The larger tube covers the whole cube \( Q_s \).
Then on $Q_s$, we have
\[
\sum_{a=1}^{N_j} T_{j,a,\delta}^{-1} \geq N_j(Q_s).
\]
Also we know that the volume of $Q_s$ is $\gtrsim \delta^{-n}$ so we are done. □

**Lemma 1.12.** Suppose $Q_s$ is a cube of size
\[
s \leq \delta^{-1},
\]
then
\[
\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C(n) \cdot \delta^n \cdot \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}},
\]
where $C(n)$ is a constant depending on $n$.

**Proof.** Suppose that
\[
Q_s = \bigcup_b Q_{t,b},
\]
where each $Q_{t,b}$ is of size
\[
t \in \left[ \frac{1}{20n} \delta^{-1}, \frac{1}{10n} \delta^{-1} \right].
\]
Then we have
\[
\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} = \sum_b \int_{Q_{t,b}} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C(n) \cdot \delta^n \cdot \sum_b \int_{Q_{t,b}} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}} = C(n) \cdot \delta^n \cdot \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}}.
\]
□

Now by the same idea but just rescaling the size, we have

**Lemma 1.13.** Suppose $Q_s$ is a cube of size
\[
s \leq \delta^{-m},
\]
then
\[
\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-m-1}} \right)^{\frac{1}{n-1}} \lesssim \delta^n \cdot \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-m}} \right)^{\frac{1}{n-1}},
\]
where the constant depends on $n$. 
proof of lemma 1.9} Suppose $s \leq \delta^m$. We induct on $m$ to prove that

$$\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C(n)^m \cdot \delta^m \cdot \int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a,\delta^{-m}} \right)^{\frac{1}{n-1}},$$

where $C(n)$ is the same as in lemma 1.13.

Now on $Q_s$, we have

$$0 \leq \sum_{j=1}^{N_j} T_{j,a,\delta^{-m}} \leq N_j,$$

so by formula (3),

$$\int_{Q_s} \prod_{j=1}^{n} \left( \sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq C(n)^m \cdot \delta^m \cdot \int_{Q_s} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

$$\leq C(n)^m \cdot \delta^m \cdot s^n \cdot \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

$$\leq C(n)^m \cdot \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}.$$

Note that

$$C(n)^m \leq C(n)^{\frac{\log(s)}{-\log(\delta)}} = s^{\frac{\log(C(n))}{-\log(\delta)}},$$

so we only need to pick small enough $\delta$ such that

$$\frac{\log(C(n))}{-\log(\delta)} \leq \varepsilon$$

to finish the proof. $\square$