The problem from last time is to estimate how big is $|\sum_{a=1}^{A} e(a^k x)|$ where $e(x) := e^{2\pi i x}$. The intuition is that if $x$ is Diophantine, then there’s cancellation.

**Lemma 0.1.** For any $a_1 \leq a_2 \in \mathbb{N} \cup \{0\}$, $|\sum_{a=a_1}^{a_2} e(ax)| \leq 10\|x\|^{-1}$, where $\|x\| := \text{dist}(x, \mathbb{Z})$.

**Proof.** Write $\alpha = e(x)$ for convenience. Then

$$|\sum_{a=a_1}^{a_2} \alpha^a| = |\sum_{a=0}^{L} \alpha^a| \quad \text{since } |\alpha| = 1$$

$$= |1 - \alpha^{L+1} \alpha^{-1}| \leq 2|1 - \alpha|^{-1} \leq 10\|x\|^{-1}$$

\[\square\]

For $P : \mathbb{Z} \to \mathbb{C}$, define the first order difference $\Delta_h P(a) := P(a + h) - P(a)$. Note that if $P(a) = c_k a^k + \cdots + c_0$, i.e. $P$ is a degree $k$ polynomial, then $\Delta_h P(a) = kh c_k a^{k-1} + \text{lower order terms}$. This follows from direct expansion.

**Lemma 0.2.** If $I \subset [-A, A] \cap \mathbb{Z}$, then

$$\left| \sum_{a \in I} e(P(a)) \right|^2 \leq \sum_{|h| \leq 2A} \sum_{b \in I_h} e(\Delta_h P(b))$$

where $I_h \subset [-A, A]$.

**Proof.**

$LHS = \sum_{a,b \in I} e(P(a) - P(b)) \quad a = b + h, P(a) - P(b) = \Delta_h P(b)$

$$= \sum_{|h| \leq 2A} \sum_{b \in I_h} e(\Delta_h P(b)) \leq RHS \quad \text{triangle inequality}$$

\[\square\]
Remark. Lemma 0.2 may have big loss from the triangle inequality, but it is good because we can use the estimate of the inner sum, which has one degree lower. So we can iterate it.

**Lemma 0.3.** \( \forall j \geq 1 \),
\[
| \sum_{a \in I} e(P(a)) |^{2j} \lesssim A^{2j-j-1} \sum_{h_1, \ldots, h_j \in I_h, |h_i| \leq 2A} | \sum_{b \in I_h} e(\Delta_h^j P(b)) |
\]
where \( \Delta_h^j P(b) \) is the \( j \)-th order difference.

**Proof.** We prove by induction. The base case \( j = 1 \) is just Lemma 0.2. Assume for \( j \), we prove for \( j + 1 \) by squaring the expression for \( j \).
\[
| \sum_{a \in I} e(P(a)) |^{2j+1} \lesssim A^{2j+1-2j-2} \left( \sum_{h_1, \ldots, h_j \in I_h, |h_i| \leq 2A} | \sum_{b \in I_h} e(\Delta_h^j P(b)) | \cdot 1 \right)^2
\]
Cauchy-Schwarz \( \lesssim A^{2j+1-j-2} \sum_h | e(\Delta_h^j P(b)) |^2 \)
Lemma 0.2 \( \lesssim A^{2j+1-j-2} \sum_{h_1, \ldots, h_j, h_{j+1}} | \sum_{b \in I_h} e(\Delta_h^{j+1} P(b)) | \),
there are \( \sim A^j \) terms to sum on RHS of the first line. \( \square \)

Remark. The trivial bound \( | \sum_{a \in I} e(P(a)) | \leq 2A \) on RHS gives the trivial bound on LHS, while improvement of this gives improvement of 1 degree higher.

**Theorem 0.4 (Weyl).** If \( P(a) = c_k a^k + \cdots \), then
\[
| \sum_{a=1}^A e(P(a)) |^{2^{k-1}} \lesssim_{k, \epsilon} A^{2^{k-1-k-\epsilon}} \sum_{|h|=1} O(A^{k-1}) \min(A, \| hc_k \|^{-1}),
\]
where \( O(A^{k-1}) = 2^{k-1} k! A^{k-1} \). In particular, if \( c_k \) is Diophantine, then from last time \( \text{Avg}_{1 \leq |h| \leq O(A^{k-1})} \| hc_k \|^{-1} \lesssim A^{\epsilon} \). This yields
\[
| \sum_{a=1}^A e(P(a)) | \lesssim_{k, \epsilon} A^{1-\sigma(k)+\epsilon} \quad \text{where } \sigma(k) = \frac{1}{2k-1}
\]

**Proof.** Use Lemma 0.3. Take \( j = k - 1 \).
\[
LHS \lesssim A^{2^{k-1-k-1}} \sum_{h_1, \ldots, h_{k-1}} | \sum_{b \in I_h} e(\Delta_h^{k-1} P(b)) |
\]
Observe that $\Delta_k^{k-1} P(b)$ is a degree 1 polynomial with leading coefficient $k!h_1 \cdots h_{k-1}c_k$. Using Lemma 0.1,

$$LHS \lesssim A^{2k-1-k} \sum_{h_1, \ldots, h_{k-1}} \min(A, \|k!h_1 \cdots h_{k-1}c_k\|^{-1}).$$

Since $k!h_1 \cdots h_{k-1} \in [1, O(A^{k-1})]$, and each $h$ has $\lesssim A^\epsilon$ repetitions of form $k!h_1 \cdots h_{k-1}$, plugging in these bounds we are done. □

**Question 0.5.** How big is the number of factors of $N = p_1^{e_1} \cdots p_s^{e_s}$, i.e. $\prod_{i=1}^s (e_i + 1)$?

**Naive answer.** $N \geq 2^{\sum e_i}$ since $p_i \geq 2$. Consider the example where $e_i = 1$ for $i = 1, \ldots, s$. Then $N \geq 2^s = \# \text{factors}$, which is not a very good estimate. But probably this is the worst case (though Robert has another guess), i.e. $p_1 = 2, p_2 = 3, \ldots$ and $e_i = 1$, then $N = p_1 \cdots p_s \geq s!$ since $p_i \geq i$, and $\# \text{factors} = 2^s$. We want $2^e (s!)^\epsilon \sim (\frac{s}{e})^{s\epsilon}$ for fixed $\epsilon$ and $s \to \infty$ (by Stirling’s formula), so we need to choose $s$ such that $(\frac{s}{e})^{s\epsilon} \geq 2$.

Vinogradov gave a very different approach to estimating such sums, which gives much better estimates when $k$ is large. His method uses the fact that $c_k$ is Diophantine in a very different way, which we will explain in this lecture. Here is his estimate.

**Theorem 0.6 (Vinogradov).** If $P(a) = c_ka^k + \cdots, c_k$ Diophantine, then

$$|\sum_{a=1}^A e(P(a))| \lesssim A^{1-\sigma(k)+\epsilon} \quad \text{where} \quad \sigma(k) \geq \frac{1}{C k^2 \log k}$$

**Remark.** The best current bound, given by decoupling and Wooley independently, is $\sigma(k) \sim 1/k^2$.

To prove the theorem we need a few lemmas.

**Lemma 0.7.** Let $f(x) := \sum_{a=1}^A e(a^kx_k + \cdots + ax_1) : \mathbb{R}^k \to \mathbb{C}$, and $\phi^t(x) = (\cdots, x_{k-1} + kt x_k, x_k)$ mod $\mathbb{Z}^k$. Then $|f(x)| \sim |f(\phi^t(x))|$ and the difference in norms is $\leq 2t$. So in particular, if $T < 1/10|f(c)|$, then $|f(\phi^t(c))| \geq 1/2|f(c)|$. 

Proof. \( f \) is \( \mathbb{Z}^k \)-periodic, and \( \text{supp} \hat{f} \subset [0,A] \times \cdots \times [0,A^k] \), so \( |f| \sim \) constant on \( A^{-1} \times \cdots \times A^{-k} \)-rectangles. If \( t \ll A \), then
\[
f(x) = \sum_{a=1}^{A+t} e(a_k x_k + \cdots + a x_1)
\sim \sum_{b=1+t}^{A} e(b_k x_k + \cdots + b x_1) \text{ value change by } \leq 2t
\]
\[
= \sum_{a=1}^{A} e(x_k(a+t)^k + x_{k-1}(a+t)^{k-1} + \cdots) \quad a = b - t
\]
\[
= \sum_{a=1}^{A} e(x_ka^k + (x_{k-1} + kt)x_{k-1}^{k-1} + \cdots )
\]
\[
= f(\phi^t(x))e(\text{something } x)
\]
\[
\square
\]

**Lemma 0.8.** If \( c_k \) is Diophantine, then \( \phi^1(c), \ldots, \phi^T(c) \) are \( T^{-(1+\epsilon)} \)-separated.

**Proof.** We look at \( x_{k-1} \)-coordinate of \( \phi^t(c) = c_{k-1} + kt c_k \). We have
\[
|\phi^{t_1}(c) - \phi^{t_2}(c)| \geq |(t_1 - t_2)kc_k| \\
\geq T \cdot T^{-2-\epsilon} = T^{-1-\epsilon}.
\]
\[
\square
\]

**Corollary 0.9.** If \( c_k \) is Diophantine and \( T \leq A^{1-\epsilon} \), then the \( A^{-1} \times \cdots \times A^{-k} \)-rectangles around \( \{ \phi^t(c) \}_{t=1}^T \) are disjoints.

**Corollary 0.10.** If furthermore \( T \leq 1/10|f(c)| \), then
\[
\int_{[0,1]^k} |f|^p \gtrsim T \cdot |\\text{rectangle}||f(c)|^p \gtrsim A^{-1-\cdots-k}|f(c)|^{p+1}.
\]

Observe that
\begin{enumerate}
\item \( f(0) = A \), \( |f(x)| \sim A \) on \( A^{-1} \times \cdots \times A^{-k} \)-rectangle, so \( \int |f|^p \gtrsim A^{\frac{k(k+1)}{2}} A^p \);
\item \( \int |f|^2 = A \), so \( \int |f|^p \geq (\int |f|^2)^{\frac{p}{2}} = A^\frac{p}{2} \) by Hölder’s inequality.
\end{enumerate}

One may conjecture these are the worst cases. Indeed, we have the following.

**Conjecture 0.11** (Vinogradov). \( \int |f|^p \lesssim A^\epsilon \max(A^{-\frac{k(k+1)}{2}} A^p, A^\frac{p}{2}) \).

**Theorem 0.12** (Decoupling, Wooley). The above conjecture is true.
We can compare the two terms in the maximum. The critical value of \( p \) is \( p_c = k(k + 1) \).

**Conjecture 0.13.** If \( p \geq p_c \), then \( \int |f|^p \lesssim A^\epsilon \cdot \p^{-\frac{k(k+1)}{2}} \). So if \( x_k \) is Diophantine, \( |f(c)|^{p+1} \lesssim A^{\frac{k(k+1)}{2}} \int |f|^p \lesssim A^{p+\epsilon} \), which implies
\[
|f(c)| \lesssim A^{p+\epsilon} = A^{1 - \frac{1}{p+1} + \epsilon}
\]
(the smaller the \( p \), the better the bound).

**Remark.** Vinogradov proved the above conjecture for \( p \geq Ck^2 \log k \). The full conjecture is for \( p = k(k + 1) \).