Problem Set 9

1. Suppose that $f_n : [0,1] \to \mathbb{R}$ are measurable functions. Suppose that $\int_{[0,1]} |f_n| = 1$ for all n, and suppose that $f_n \to f$ pointwise. What are the possible values of $\int_{[0,1]} |f|$? Explain your answer.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an L^1 function. In other words, suppose that f is measurable and that $\int_{\mathbb{R}} |f| < \infty$. For all $x \geq 0$, define $F(x)$ as the integral

$$F(x) := \int_{[0,x]} f.$$

Prove that F is continuous.

In the next problems, we make some connections between Lebesgue integration and Fourier analysis.

3. The Lebesgue dominated convegence theorem makes it much easier to understand the derivative of an integral, which makes the Fourier analysis approach to PDE technically smoother. For example, consider the Schrodinger equation $\partial_t u(x,t) = i\partial_x^2 u(x,t)$ for a function $u(x,t)$, where $x \in \mathbb{R}$ and $t \geq 0$, with initial data $u(x,0) = f(x)$ for $f \in \mathcal{S}$. We gave the following formula for the solution

$$u(x,t) := \int_{\mathbb{R}} e^{i(2\pi i\omega)^2 t} e^{2\pi i\omega x} \hat{f}(\omega) d\omega.$$

Once we are inspired by Fourier analysis to guess that this formula solves the differential equation, we just have to check that it gives the right solution by computing $\partial_t u$ and $\partial_x^2 u$. Using Lebesgue dominated convergence, prove that

$$\partial_t \left[\int_{\mathbb{R}} e^{i(2\pi i\omega)^2 t} e^{2\pi i\omega x} \hat{f}(\omega) d\omega \right] = \int_{\mathbb{R}} \partial_t \left(e^{i(2\pi i\omega)^2 t} \right) e^{2\pi i\omega x} \hat{f}(\omega) d\omega.$$

(By the same argument, you can also check that you can bring the ∂_x and ∂_x^2 inside the integral. You don't have to write it up, but on your own check that we also have:

$$\partial_x^2 \left[\int_{\mathbb{R}} e^{i(2\pi i\omega)^2 t} e^{2\pi i\omega x} \hat{f}(\omega) d\omega \right] = \int_{\mathbb{R}} e^{i(2\pi i\omega)^2 t} \partial_x^2 \left(e^{2\pi i\omega x} \right) \hat{f}(\omega) d\omega.$$

With that analysis done, a direct computation quickly shows that $\partial_t u(x,t) = i\partial_x^2 u(x,t)$. Other PDE can be dealt with in the same way.)

4. Suppose that a_n are complex numbers with $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$. Let $f_N := \sum_{|n|\leq N} a_n e^{inx}$.

a.) Prove that f_N is a Cauchy sequence in $L^1([0,2\pi])$. This means that for any $\epsilon > 0$, there is some large N so that if $N_1, N_2 \geq N$, then

$$\int_{[0,2\pi]} |f_{N_1} - f_{N_2}| < \epsilon.$$

By the Riesz-Fischer theorem (Theorem 2.2 on page 70 in the Real Analysis book), the space L^1 is complete, and so every Cauchy sequence has a limit in L^1. In particular, there is a function $f_\infty \in L^1([0,2\pi])$ so that

$$\int_{[0,2\pi]} |f_N - f_\infty| \to 0.$$

The definition of the Fourier coefficient $\hat{f}(n)$ makes sense when f is an L^1 function from $[0,2\pi]$ to $\mathbb{C}$. If $f \in L^1([0,2\pi])$, then we define

$$\hat{f}(n) := \frac{1}{2\pi} \int_{[0,2\pi]} f(x) e^{-inx} dx.$$

b.) With a_n, f_N, and f_∞ as above, prove that $\hat{f}_\infty(n) = a_n$.