Problem Set 8

1. For a set $E \subset \mathbb{R}^d$, we define the distance from x to E by $d(x, E) := \inf_{y \in E} |x - y|$. We define the r-neighborhood of E, $N_r(E)$, by

$$N_r(E) := \{x \in \mathbb{R}^d | d(x, E) < r\}.$$

If $E \subset \mathbb{R}^d$ is compact, prove that

$$\lim_{r \to 0} m_*(N_r(E)) = m_*(E). \qquad (*)$$

2. (Followup on Question 1.) Construct a closed set E where $(*)$ fails.
Construct a bounded set E where $(*)$ fails.
Construct an open set E where $(*)$ fails.

3. Find a countable collection of disjoint closed intervals $I_j \subset [0, 1]$ with $\sum_{j=1}^{\infty} |I_j| = 1$.

4. Suppose that $A \subset \mathbb{R}$ is a measurable set with measure 1. Let $A + t$ be the translation of A by t: $A + t := \{a + t | a \in A\}$. Prove that

$$\lim_{t \to 0} m_*(A \cap A + t) = 1?$$

Hint: Approximate the set A by a simpler set.

In the next section of the course, we will define the Lebesgue integral, and we will prove several convergence theorems about interchanging limits and integrals. To get ready for this, we will prove analogous convergence theorems for series. These are elementary, but they also give good intuition for the harder case of integrals. The basic question is the following. We are interested in a sum of the form $\sum_{j=1}^{\infty} a(j)$. We have a sequence of sums $\sum_{j=1}^{\infty} a_n(j)$, and we for each j, we know that $\lim_{n \to \infty} a_n(j) = a(j)$. How much more information do we need in order to be sure that

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_n(j) = \sum_{j=1}^{\infty} a(j)?$$

5a. (Non-convergence) Give an example where $a_n(j) \to a(j)$ for each j, and $\sum_{j=1}^{\infty} a_n(j)$, $\sum_{j=1}^{\infty} a(j)$ all converge absolutely, but

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_n(j) \neq \sum_{j=1}^{\infty} a(j).$$

5b. (Dominated convergence) Suppose that for all j, n, $|a_n(j)| \le b(j)$, and that $\sum_{j=1}^{\infty} b(j) < \infty$. If $a_n(j) \to a(j)$ for each j, then prove that

$$\lim_{n \to \infty} \sum_{j=1}^{\infty} a_n(j) = \sum_{j=1}^{\infty} a(j).$$