Problem Set 7

1. (A variant on the definition of exterior measure). For a cube $Q \subset \mathbb{R}^d$ of side length $S > 0$, define

$$C_r(Q) := S^r.$$

For a subset $E \subset \mathbb{R}^d$, define

$$M_r(E) := \inf_{E \subset \cup_{j=1}^{\infty} Q_j} \sum_{j=1}^{\infty} C_r(Q).$$

If $r = d$, them M_r is exactly the exterior measure m_*.
If $r > d$, show that $M_r([0,1]^d) = 0$.
If $0 < r < d$, show that $M_r([0,1]^d) = 1$.

2. If $m_*(E) = 0$, prove that E is a measurable set.

3. Suppose that $E \subset \mathbb{R}^d$ is any set and $\epsilon > 0$. Prove that there is an open set O containing E so that $m_*(O) \leq m_*(E) + \epsilon$.

4. Recall that $E \triangle F$ denotes the symmetric difference of E and F – the set of points that lie in exactly one of E, F. Suppose that $E \subset \mathbb{R}^d$ is well-approximated by measurable sets in the following sense: for any $\epsilon > 0$, there is a measurable set F so that $m_*(E \triangle F) < \epsilon$. Prove that E is measurable.

5. Let $I \subset \mathbb{R}$ denote the set of irrational numbers. Find a way to decompose I into two sets, C and S, so that C is closed and $m_*(S) < 1/10$. (The letter C stands for closed, and the letter S stands for small. By a decomposition, I mean that C and S are disjoint sets with union I.)

(As we will learn on Friday, any measurable set can be decomposed into a closed set and a small set. More formally, if E is measurable and $\epsilon > 0$, then $E = C \cup S$ with C closed, $m_*(S) < \epsilon$ and C and S disjoint.)