

Problem Set 6: Partial differential equations and Fourier analysis

The Fourier transform gives a systematic approach to study a wide variety of partial differential equations, and to understand the different properties of different equations. In class, we studied the heat equation. In this problem set, you will adapt these ideas to study other PDE.

Background from class. We recall here some background from the lectures that will help us apply Fourier theory to partial differential equations. The basic fact that makes this connection possible is a simple formula for the Fourier transform of a derivative.

Proposition 0.1. *If $f \in \mathcal{S}$, and $g = f'$, then*

$$\hat{g}(\omega) = 2\pi i \omega \hat{f}(\omega).$$

Solutions to a partial differential equations depend on a space variable x and a time variable t . There is a fairly natural generalization of Schwartz functions to this context, which we call the space $\mathcal{SU}$. Here is the definition. Suppose that $u : \mathbb{R} \times I \rightarrow \mathbb{C}$ is a function, where $I \subset \mathbb{R}$ is an interval (which may be open or closed at each end). For instance, I may be $[0, +\infty)$ or $[0, T]$. We say that the function $u \in \mathcal{SU}(\mathbb{R} \times I)$ if u is C^∞ (in both x and t), and if, for any $k, l, m \geq 0$, and any compact interval $I' \subset I$,

$$\sup_{t \in I'} \sup_{x \in \mathbb{R}} |x|^k |\partial_x^l \partial_t^m u(x, t)| < +\infty.$$

Informally, any mixed derivative $|\partial_x^l \partial_t^m u|$ decays very rapidly in the x -variable, and this decay is uniform over any compact set of times t .

We write $\hat{u}(\omega, t)$ for the Fourier transform of u in the x variable:

$$\hat{u}(\omega, t) := \int_{\mathbb{R}} u(x, t) e^{-2\pi i \omega x} dx.$$

Similarly, if v is a function of ω, t , we write $\check{v}$ for the inverse Fourier transform in the ω variable:

$$\check{v}(x, t) := \int_{\mathbb{R}} v(\omega, t) e^{2\pi i \omega x} d\omega.$$

Here are two fundamental properties of the space $\mathcal{SU}$.

Proposition 0.2. *If $u \in \mathcal{SU}(\mathbb{R} \times I)$, then $\hat{u} \in \mathcal{SU}(\mathbb{R} \times I)$.*

The proof of this proposition is the same as the proof that if $f \in \mathcal{S}$, then $\hat{f} \in \mathcal{S}$ – Theorem 1.3 in Chapter 5 of the book.

Proposition 0.3. *If $u(x, t) \in \mathcal{SU}(\mathbb{R} \times I)$, then $\partial_t \hat{u}(\omega, t) = (\partial_t u)^\wedge(\omega, t)$. Similarly, if $v(\omega, t) \in \mathcal{SU}(\mathbb{R} \times I)$, then $\partial_t \check{v}(x, t) = (\partial_t v)^\vee(x, t)$.*

We will prove Proposition 0.3 in the last problem. Other than that, you can use both Propositions freely.

The first PDE that we study is

$$(1) \quad \partial_t u(x, t) = -\partial_x^4 u(x, t)$$

1. If $f \in \mathcal{S}$, then prove that the following function $u(x, t)$ is in $\mathcal{SU}(\mathbb{R} \times [0, \infty))$ and obeys $u(x, 0) = f(x)$ and $\partial_t u = \partial_x^4 u$:

$$u(x, t) = \int_{\mathbb{R}} e^{2\pi i \omega x} e^{-(2\pi)^4 \omega^4 t} \hat{f}(\omega) d\omega.$$

(This function u is the unique solution $u \in \mathcal{SU}$ that obeys the initial condition $u(x, 0) = f(x)$ and $\partial_t u = \partial_x^4 u$. It is not difficult to prove this, but you don't have to do so on the problem set.)

We study this solution $u(x, t)$ over the next four problems.

2. Show that $\int_{\mathbb{R}} |u(x, t)|^2 dx$ is a non-increasing function of t .

3. Show $\sup_x |u(x, 1)| \leq 100 \int_{\mathbb{R}} |f(x)| dx$.

4. (Smoothing estimate) Show that $\sup_x |\partial_x u(x, 1)| \leq 1000 \int_{\mathbb{R}} |f(x)| dx$.

5. (Decay estimate) For any $t > 0$, show that $\sup_x |u(x, t)| \leq 1000 t^{-1/4} \int_{\mathbb{R}} |f(x)| dx$.

6. Next we study the partial differential equation $\partial_t u(x, t) = \partial_x^4 u(x, t)$. This equation is quite badly behaved. Even with very nice initial data, this equation often has no solution. Prove that there exists a function $f \in \mathcal{S}$ so that, for every $T > 0$, there is no function $u \in \mathcal{SU}(\mathbb{R} \times [0, T])$ obeying $u(x, 0) = f(x)$ and $\partial_t u = \partial_x^4 u$.

7. Suppose $v \in \mathcal{S}$. Prove that $e^{i\omega^2} v(\omega)$ is a Schwartz function also.

Finally we study the partial differential equation $\partial_t u(x, t) = i\partial_x^2 u(x, t)$, the Schrodinger equation. This equation plays a fundamental role in quantum mechanics. Given initial data $f \in \mathcal{S}$, there is a solution $u \in \mathcal{SU}(\mathbb{R} \times \mathbb{R})$ defined by the equation

$$u(x, t) = \int_{\mathbb{R}} e^{2\pi i \omega x} e^{-i(2\pi)^2 \omega^2 t} \hat{f}(\omega) d\omega.$$

The proof that this function u solves the partial differential equation $\partial_t u(x, t) = i\partial_x^2 u(x, t)$ is very similar to Problem 1. The one point which is a bit trickier is to check that the functions involved are in $\mathcal{SU}$, and the main issue is dealt with in

Problem 7. In the next two problems we prove a couple properties of the solution $u(x, t)$.

8. For this solution u , prove that $\int_{\mathbb{R}} |u(x, t)|^2 dx$ is constant in time.

9. For this solution u , prove that $\int_{\mathbb{R}} |\partial_x u(x, t)|^2 dx$ is constant in time also.

If you are interested in partial differential equations, it is a very nice problem to try to understand whether solutions to the Schrodinger equation decay in time. This is a little beyond the level of this problem set, but I would be happy to talk about it with you. For instance, we might suppose that the initial data f has compact support in $[-1, 1]$, and that $|f|, |\partial_x f|, \dots, |\partial_x^5 f|$ are all bounded by 1. Under these circumstances, how does $|u(0, t)|$ behave for large t ? Does it go to zero? If so, at what rate?

10. Prove Proposition 0.3. Here is a little bit of outline/hint about exchanging a derivative and an integral. By definition of a derivative, we want to show that

$$\lim_{h \rightarrow 0} \frac{1}{h} \left[\int_{\mathbb{R}} v(\omega, t+h) e^{2\pi i \omega x} d\omega - \int_{\mathbb{R}} v(\omega, t) e^{2\pi i \omega x} d\omega \right] = \int_{\mathbb{R}} \partial_t v(\omega, t) e^{2\pi i \omega x} d\omega.$$

In other words, we are trying to prove that when h is really small, the left-hand side and the right-hand side are close together. So this is a problem about estimating their difference. We want to control the size of

$$\left| \frac{1}{h} \int_{\mathbb{R}} v(\omega, t+h) e^{2\pi i \omega x} d\omega - \frac{1}{h} \int_{\mathbb{R}} v(\omega, t) e^{2\pi i \omega x} d\omega - \int_{\mathbb{R}} \partial_t v(\omega, t) e^{2\pi i \omega x} d\omega \right|.$$

At this point, the problem becomes like any other estimate. We know that $v \in \mathcal{SU}$, which gives us a lot of estimates about v , and we want to use those estimates to control the expression above. Finding a good way to group the terms is a good first step...