

Problem Set 5

Background reading: A review of Fubini's theorem.

When we start to study the Fourier transform, we will frequently use Fubini's theorem. Here is a quick review of the Fubini theorem for continuous functions for your reference. Just read it over and feel free to use it as a tool.

On a product of closed intervals, the statement of Fubini's theorem is simple:

Theorem 1. *Suppose that $F(x, y)$ is a continuous function on $[a, b] \times [c, d]$. Then*

$$\int_{[a,b] \times [c,d]} F(x, y) dx dy = \int_a^b \left(\int_c^d F(x, y) dy \right) dx = \int_c^d \left(\int_a^b F(x, y) dx \right) dy.$$

On \mathbb{R}^2 , the situation is slightly more complicated. Recall that in Riemann integration, $\int_{\mathbb{R}} F(x) dx := \lim_{R \rightarrow \infty} \int_{-R}^R F(x) dx$. The following result is a corollary of Theorem 1.

Theorem 2. *Suppose that $F(x, y)$ is a continuous function on \mathbb{R}^2 . Suppose that*

$$\int_{\mathbb{R}^2} |F(x, y)| dx dy := \lim_{R \rightarrow \infty} \int_{[-R, R]^2} |F(x, y)| dx dy < \infty.$$

Also suppose that $\int_{\mathbb{R}} F(x, y) dy$ is continuous in x and $\int_{\mathbb{R}} F(x, y) dx$ is continuous in y . Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x, y) dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F(x, y) dy \right) dx.$$

Moreover, either of these expressions is equal to

$$\lim_{R_1, R_2 \rightarrow \infty} \int_{[-R_1, R_1] \times [-R_2, R_2]} F(x, y) dx dy.$$

You can freely use either version of Fubini's theorem in your work. Later in the course, we will discuss the Fubini theorem using the Lebesgue integral, and we will be able to get rid of the minor condition that $\int F(x, y) dx$ is continuous. A small remark: if $F(x, y)$ is continuous on \mathbb{R}^2 , and $\int_{\mathbb{R}^2} |F(x, y)| dx dy < \infty$, it is not necessarily the case that $G(y) := \int F(x, y) dx$ is Riemann integrable. To use Fubini in such a tricky situation requires the Lebesgue theory. We will talk about this more in November.

1. (Using Cauchy-Schwarz)

a.) Give an example of a sequence a_n so that $\sum_{n=1}^{\infty} |a_n|^2$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

b.) Suppose that $\sum_{n=1}^{\infty} |a_n|^2 n^2 < \infty$. Show that $\sum_{n=1}^{\infty} |a_n| < \infty$.

Hint: Apply the Cauchy-Schwarz inequality to $\sum_{n=1}^{\infty} |a_n|$ in a tricky way, so that $\sum_{n=1}^{\infty} |a_n|^2 n^2$ appears on one side of the inequality.

2. (On convergence of Fourier series) We proved in class that if f is C^2 and 2π periodic, then the partial sums $S_N f$ converge to f uniformly. This result also holds if f is only C^1 and 2π periodic.

If f is C^1 and 2π periodic, prove that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$.

Hint: Use Problem 1b.

Remark: Since f is C^0 and 2π -periodic and $\sum_n |\hat{f}(n)| < \infty$, it follows by a theorem from class that $S_N f$ converges to f uniformly. (This theorem appears in the book in Chapter 2.2.)

3. (A variation of the isoperimetric inequality) Suppose that $\Gamma \subset \mathbb{R}^2$ is a simple closed curve and that $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$ is a parametrization of Γ . Suppose $\gamma(s) = (x(s), y(s))$ where $x(s)$ and $y(s)$ are C^2 2π -periodic functions.

Suppose that $|x'(s)| \leq 1$ for all s and that $\int_0^{2\pi} |y(s)|^2 ds \leq 1$. Under these conditions, the length of Γ may be arbitrarily large. Nevertheless, prove that the area enclosed by Γ is at most 100.

4. Weyl equidistribution criterion.

Recall that if x is a real number then $\langle x \rangle \in [0, 1)$ is the fractional part of x . In class, we proved that if γ is an irrational number, then the sequence $\langle n\gamma \rangle$ is equidistributed in the unit interval. More generally, the same idea leads to Weyl's equidistribution criterion:

Theorem 3. (Weyl) *Suppose that z_n is a sequence of real numbers. Then $\langle z_n \rangle$ is equidistributed in the unit interval if and only if, for each non-zero $k \in \mathbb{Z}$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i k z_n} = 0.$$

Write a 4-6 sentence outline of the proof of the Weyl equidistribution criterion. It makes the notation a little shorter to define

$$e(x) := e^{2\pi i x}.$$

We will use this notation in the next problem.

5. In this problem, we will apply the equidistribution criterion to the sequence $\sqrt{2}n^2$.

Prove that

$$\sum_{n=1}^N e(\sqrt{2}n^2) \leq 1000N^\alpha,$$

for some exponent $\alpha < 1$. More generally, prove that for any non-zero $k \in \mathbb{Z}$,

$$\sum_{n=1}^N e(k\sqrt{2}n^2) \leq C(k)N^\alpha.$$

This estimate together with the equidistribution criterion implies that the numbers $\langle \sqrt{2}n^2 \rangle$ are equidistributed in the unit interval.

Hints: This is the most challenging problem we have worked on so far in the course. The first hint is to imitate the Gauss sum estimate that we proved in class. The second hint is that you will need an estimate for how close a rational number is to $\sqrt{2}$. As a lemma, you might want to prove that

$$\left| \sqrt{2} - \frac{p}{q} \right| \geq \frac{1}{100}q^{-2}.$$

Finally, feel free to come to office hours.