Problem Set 4

Hamel vs Hilbert bases

In the next two problems we will explore two different notions of a basis for a Hilbert space.

Definition: Let V be a vector space over $\mathbb{C}$. We say a set $B \subset V$ is a (Hamel) basis if the elements of B are linearly independent, and every element of V can be written as a finite linear combination of elements from B.

Definition: Let V be a Hilbert space over $\mathbb{C}$. We say that a set $B \subset V$ is a (Hilbert) basis if the elements of B are orthonormal, and for every $v \in V$ there exist sequences $\{w_i\} \subset V$ and $\{c_i\} \subset \mathbb{C}$ so that $\left\| v - \sum_{i=1}^{N} c_i w_i \right\| \to 0$ as $N \to \infty$.

Prove each of the following statements, or provide a counter-example.

1. Let V be a finite dimensional Hilbert space over $\mathbb{C}$. A) Every Hamel basis for V is a Hilbert basis. B) Every Hilbert basis is a Hamel basis.

2. Let V be a Hilbert space over $\mathbb{C}$ (possibly infinite dimensional). A) every Hamel basis for V is a Hilbert basis. B) Every Hilbert basis is a Hamel basis.

Using Fourier series to evaluate $\sum 1/n^2$.

3. Consider the function $f(x) = x$ for $x \in (-\pi, \pi)$. Since f is continuous and bounded, we know that f can be written in the form $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-inx}$. Compute the terms $\{a_n\}$ (i.e. compute the Fourier series of f).

4. Using Parseval's identity (discussed in class, or see p79 of Stein-Shakarchi) and the computation of $\{a_n\}$ from Problem 3, evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The following exercises are from Stein-Shakarchi.

5. Suppose that f is 2π periodic and of class C^k (i.e. the first k derivatives of f exist, and $f^{(k)}$ is continuous). Show that $|n|^k \hat{f}(n) \to 0$ as $n \to \infty$. This is an example of a more general phenomena—the smoothness of a function is closely linked to the decay of its Fourier transform.

6. Let f be a 2π periodic C^1 function with $\int_0^{2\pi} f(t)dt = 0$. Show that

$$\int_0^{2\pi} |f(t)|^2 dt \leq \int_0^{2\pi} |f'(t)|^2 dt.$$

There are a broad class of results like the one above; they are known as Poincaré inequalities.