Problem Set 3

1. Suppose that the closed interval $[a,b]$ is contained in the open interval $(-\pi,\pi)$. Given any continuous function $f:[a,b]\to\mathbb{C}$, and any $\epsilon>0$, prove that there is a trigonometric polynomial $P(x)=\sum_{n=-N}^{N}c_n e^{inx}$ so that $|P(x)-f(x)|<\epsilon$ on $[a,b]$.

Using this result, prove that given any continuous function $f:[a,b]\to\mathbb{R}$, for any $\epsilon>0$, there is a polynomial $Q(x)=\sum_{n=0}^{N}d_n x^n$ so that $|Q(x)-f(x)|<\epsilon$ on $[a,b]$. This is called the Weierstrauss approximation theorem.

In the next couple problems, we will explore some differences between partial sums and Fejer sums of Fourier series.

2. Suppose that f is C^0 and 2π-periodic, with $f(x)=0$ for $0\le x\le\pi$ and $|f(x)|\le 1$ for all x. Prove that

$$|\sigma_N f(\pi/2)|\le 100N^{-1}.$$

3. The situation is different for the partial sums of f. For any $N\ge 1$, show that there exists a C^0, 2π-periodic function f_N, so that $f_N(x)=0$ for $0\le x\le\pi$ and $|f_N(x)|\le 1$ for all x, and yet

$$|S_N f_N(\pi/2)|\ge\frac{1}{100}.$$

Analysis cobwebs: interchanging a derivative and an integral. Over the week, we talked about interchanging a limit and an integral, and this is a related topic.

Suppose that $g(x,y)$ is a function of x,y which is differentiable in x. Let

$$G(x)=\int_a^b g(x,y)dy.$$

Suppose we want to understand

$$\frac{d}{dx}G(x)=\frac{d}{dx}\left(\int_a^b g(x,y)dy\right).$$

When can we put the derivative inside the integral? In other words, when can we write

$$\frac{d}{dx}\left(\int_a^b g(x,y)dy\right)=\int_a^b\frac{\partial}{\partial x}g(x,y)dy.$$

A good approach to this question is to write the left-hand side as a limit:

$$\frac{d}{dx}\left(\int_a^b g(x,y)dy\right) = \lim_{h\to 0}\frac{\int_a^b g(x+h,y)dy - \int_a^b g(x,y)dy}{h} = \lim_{h\to 0}\int_a^b \frac{g(x+h,y)-g(x,y)}{h}dy.$$

Now the question reduces to interchanging a limit and an integral. The following problem involves this situation.

4. Suppose that f is C^0 and 2π-periodic and g is C^1 and 2π-periodic. Let $h = f * g$. Prove carefully that

$$\frac{dh}{dx} = f * \frac{dg}{dx}.$$

5. (Very good kernels)

Let us say that a family K_N is a very good family of kernels if it is a good family of kernels and in addition, each function K_N is C^1, and for any $\delta > 0$,

$$\int_{\delta\le |x|\le \pi} |K_N'(x)| dx \to 0 \text{ as } N \to \infty.$$

As in Problems 2 and 3, suppose that f is C^0 and 2π-periodic, with $f(x) = 0$ for $0 \le x \le \pi$. Suppose that K_N is a very good family of kernels. Let $g_N := f * K_N$.

Prove that $g_N'(\pi/2) \to 0$.

6. Neither the Dirichlet kernel nor the Fejer kernel is a very good family of kernels. In the last pset, we introduced another family of kernels. For integers $N \ge 1$, let $I_N := \frac{1}{2\pi}\int_0^{2\pi}(1+\cos x)^N dx$. Define $K_N = (1+\cos x)^N / I_N$. The kernel K_N is a trigonometric polynomial of degree N. On the last pset, you proved that K_N is a good family of kernels. Now prove the additional properties to check that K_N is a very good family of kernels.

Extra credit: Suppose that f is a C^0 2π-periodic function, and suppose that f is C^1 on a neighborhood of x_0. Suppose that K_N is a very good family of kernels. Let $g_N = f * K_N$. Prove that $g_N'(x_0) \to f'(x_0)$.