Problem Set 1

Reading: Read the course description, especially the discussion of the different applications of Fourier analysis.

Problems:

1. Let $f(x)$ be defined as follows: if $0 \leq x < \pi$, then $f(x) = 1$; if $\pi \leq x < 2\pi$, then $f(x) = 0$; f extends to be 2π-periodic on $\mathbb{R}$. (This is the function that we discussed in the first lecture.)

Compute the Fourier coefficients $\hat{f}(n)$. (Recall $\hat{f}(n) := (1/2\pi) \int_0^{2\pi} f(x)e^{-inx}dx$.)

Analysis cobwebs: From time to time in the psets, we will review important ideas from 18.100. In this pset, we review uniform convergence and continuity.

Recall that a sequence of functions f_n converges uniformly to a function f if for every $\epsilon > 0$, there exists N so that for all $n \geq N$ and all x, $|f_n(x) - f(x)| < \epsilon$. In the first analysis course, you learned the following proposition:

Proposition 1. *If $f_n : \mathbb{R} \to \mathbb{C}$ are continuous functions and f_n converges uniformly to f, then f is also continuous.*

Read over the following proof. Some of the exercises below are based on this proposition and/or its proof.

Proof. For any $x \in \mathbb{R}$ and any $\epsilon > 0$, we have to find some $\delta > 0$ and we have to prove that whenever $|x - x'| < \delta$, $|f(x) - f(x')| < \epsilon$.

By uniform continuity, there exists an N so that for all $x' \in \mathbb{R}$,

$$|f_N(x') - f(x')| < \frac{\epsilon}{4}. \tag{1}$$

Since f_N is continuous, there exists a $\delta > 0$ so that whenever $|x - x'| < \delta$,

$$|f_N(x') - f_N(x)| < \frac{\epsilon}{4}. \tag{2}$$

Now, whenever $|x - x'| < \delta$, we can bound $|f(x) - f(x')|$ using (1) and (2):

$$|f(x) - f(x')| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$

$\square$

2. Consider a trigonometric series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$. In this sum, n ranges over all the integers, and for each n, a_n is a complex number. Suppose that $\sum_n |a_n|$ is finite.

This implies that for each $x \in \mathbb{R}$, the sum converges. Therefore, the sum defines a function $f : \mathbb{R} \to \mathbb{C}$,

$$f(x) := \sum_{n=-\infty}^{\infty} a_n e^{inx}.$$

Prove that f is continuous.

3. Let f be as in Problem 2: $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, where $\sum_n |a_n|$ is finite. Recall that $\hat{f}(n)$ is defined as
$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$
Prove that $\hat{f}(n) = a_n$.
This exercise is a step in our proof that Fourier series converge.

4. One of the themes in this course is proving quantitative estimates. In a first course on real analysis, we might prove that a function is continuous. Saying that a function is continuous is a qualitative statement about the function. In this course, we will often be interested in quantitative statements. To start to get a feel for going from qualitative statements to quantitative statements, we now consider a quantitative problem related to Proposition 1.
Suppose that $f_n : [0,1] \to \mathbb{C}$ and $f : [0,1] \to \mathbb{C}$ are functions that obey the following inequalities. Suppose that

$$|f_n(x) - f(x)| < \frac{1}{n}. \tag{1}$$

Also suppose that each f_n obeys the following estimate: for any $x, x' \in [0,1]$,

$$|f_n(x) - f_n(x')| \le n|x - x'|. \tag{2}$$

The first inequality implies that f_n converges to f uniformly and the second inequality implies that each function f_n is continuous. By Proposition 1, it follows that f is continuous. But in this problem, we are interested in quantitative estimates about the continuity of f. Prove that the limiting function f obeys the following estimate:

$$|f(x) - f(x')| \le 1000|x - x'|^{1/2}.$$

Introductory Letter:

Write me an email introducing yourself. (My email address is lguth@math.mit.edu)
What interested you in taking this course? What other math courses have you taken (and what are you taking now)?

In particular, tell me about the analysis classes you've taken. What parts or topics did you find interesting? Were there parts that you found boring or frustrating?

Are there particular topics or skills you'd like to work on this semester? Is there anything else you'd like me to know?