Practice Midterm Solutions

1. Suppose that f is continuous and 2π-periodic. Is it true that

$$\lim_{N\to\infty}\int_0^{2\pi}|S_Nf(x)-f(x)|dx=0?$$

Explain your answer.

Yes. We know from a theorem in the course that $\|S_Nf-f\|\to 0$, and so

$$\int_0^{2\pi}|S_Nf-f|^2\to 0.$$

Now by Cauchy-Schwarz

$$\int_0^{2\pi}|S_Nf(x)-f(x)|dx=\int_0^{2\pi}|S_Nf(x)-f(x)|\cdot 1dx\le$$

$$\le\left(\int_0^{2\pi}|S_Nf(x)-f(x)|^2dx\right)^{1/2}(2\pi)^{1/2}\to 0.$$

2. Let g_N be the 2π-periodic function defined on $[-\pi,\pi]$ by setting $g_N(x)=\pi N$ if $|x|\le 1/N$ and $g_N(x)=1/N$ if $1/N\le|x|\le\pi$. Suppose that f is a C^0 and 2π-periodic function. Prove from first principles that $\lim_{N\to\infty}f*g_N(0)=f(0)$.

"Prove from first principles" means that you cannot cite the good kernel theorem, but you can imitate the proof of the good kernel theorem.

By the definition of a convolution,

$$f*g_N(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)g_N(-y)dy=\frac{1}{2\pi}\int_{-1/N}^{1/N}\pi Nf(y)dy+\frac{1}{2\pi}\int_{1/N\le|y|\le\pi}(1/N)f(y)dy.$$

We will show that the first integral tends to $f(0)$ and that the second integral tends to 0.

Since f is C^0, for any $\epsilon>0$, there is an N_ϵ so that for all $N\ge N_\epsilon$, for all $|y|\le 1/N$, $|f(0)-f(y)|<\epsilon$. We also note that $\frac{1}{2\pi}\int_{-1/N}^{1/N}\pi N=1$. So if $N>N_\epsilon$, we see that

$$\left|\frac{1}{2\pi}\int_{-1/N}^{1/N}\pi Nf(y)dy-f(0)\right|=\left|\frac{1}{2\pi}\int_{-1/N}^{1/N}\pi N\left(f(y)-f(0)\right)dy\right|\le$$

$$\le\frac{1}{2\pi}\int_{-1/N}^{1/N}\pi N\epsilon dy=\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this shows that $\frac{1}{2\pi}\int_{-1/N}^{1/N} \pi N f(y)dy \to f(0)$.
On the other hand,

$$\left|\frac{1}{2\pi}\int_{1/N \le |y| \le \pi} (1/N)f(y)dy\right| \le \max|f| \cdot (1/N) \to 0.$$

3. This problem is about the partial differential equation $\partial_t u(x,t) = \partial_x^3 u(x,t)$ (where $x \in \mathbb{R}$).

a.) Consider the initial value problem $\partial_t u(x,t) = \partial_x^3 u(x,t)$, with initial data $u(x,0) = f(x)$, where f is a Schwartz function. Give a formula for the solution $u(x,t)$ in terms of $\hat{f}$. You don't need to give a full proof that the formula holds.

Solution to a.)
First we rewrite the equation in terms of $\hat{u}$:

$$\partial_t \hat{u}(\omega,t) = (2\pi i\omega)^3 \hat{u}(\omega,t).$$

We have the initial values $\hat{u}(\omega,0) = \hat{f}(\omega)$. Solving the resulting ODE gives

$$\hat{u}(\omega,t) = e^{(2\pi i\omega)^3 t}\hat{f}(\omega) = e^{-8\pi^3 i\omega^3 t}\hat{f}(\omega).$$

Now Fourier inversion gives

$$u(x,t) = \int_{\mathbb{R}} e^{2\pi i\omega x} e^{-8\pi^3 i\omega^3 t}\hat{f}(\omega)d\omega. \qquad (*)$$

b.) Using the formula from part a.), prove the following estimate. Suppose that

- $\int_{\mathbb{R}} |f(x)|dx \le 1$,
- $\int_{\mathbb{R}} |\partial_x^2 f(x)|dx \le 1$.

Prove that $|u(x,t)| \le 100$ for all $x \in \mathbb{R}$ and all $t \ge 0$.

Solution to b.): First we use our two inequalities about f to estimate $\hat{f}(\omega)$. For any function g, the triangle inequality gives

$$|\hat{g}(\omega)| = \left|\int_{\mathbb{R}} e^{-2\pi i\omega x} g(x)dx\right| \le \int_{\mathbb{R}} |g(x)|dx.$$

In particular, the first inequality gives immediately $|\hat{f}(\omega)| \le 1$ for all $\omega \in \mathbb{R}$.

Let $f_2 := \partial_x^2 f$. As on the formula sheet, we know that $\hat{f}_2(\omega) = (2\pi i\omega)^2 \hat{f}(\omega)$. Since $\int_{\mathbb{R}} |f_2(x)|dx \le 1$, we see that

$$\left|(2\pi i\omega)^2 \hat{f}(\omega)\right| \le 1,$$

and so

$$|\hat{f}(\omega)| \leq \frac{1}{4\pi^2\omega^2}.$$

Combining our two bounds on $\hat{f}(\omega)$, we see that

$$|\hat{f}(\omega)| \leq \min\left(1, \frac{1}{4\pi^2\omega^2}\right) \leq \frac{10}{1+|\omega|^2}.$$

Finally, we insert this information into the formula $(*)$ for the solution u:

$$|u(x,t)| = \left| \int_{\mathbb{R}} e^{2\pi i\omega x} e^{-8\pi^3 i\omega^3 t} \hat{f}(\omega) d\omega \right| \leq \int_{\mathbb{R}} |\hat{f}(\omega)| d\omega \leq$$

$$\leq \int_{\mathbb{R}} \frac{10}{1+|\omega|^2} d\omega \leq 100.$$

4. Suppose that $g(x) = 1 + \cos x$. In this problem, we consider what happens when we convolve g with itself many times. Here we use convolution in the setting of periodic functions, so

$$f_1 * f_2(x) := \frac{1}{2\pi} \int_0^{2\pi} f_1(y) f_2(x-y) dy.$$

Let $g_2 := g * g$. Let $g_{k+1} := g * g_k$. Find $\lim_{k\to\infty} g_k(1)$.

Hint: Recall that if $f_3 = f_1 * f_2$, then $\hat{f}_3(n) = \hat{f}_1(n)\hat{f}_2(n)$.

We know that $\hat{g}_k(n) = (\hat{g}(n))^k$. Now $g(x) = 1 + \cos x = 1 + (1/2)e^{ix} + (1/2)e^{-ix}$. Therefore, $\hat{g}(0) = 1$, $\hat{g}(\pm 1) = (1/2)$ and $\hat{g}(n) = 0$ for other n. Therefore, $\hat{g}_k(0) = 1$, $\hat{g}_k(\pm 1) = 2^{-k}$, and $\hat{g}(n) = 0$ for other n. Since g_k is clearly smooth, it is equal to the sum of its Fourier series, and we get the simple formula

$$g_k = 1 + 2^{-k} e^{ix} + 2^{-k} e^{-ix} = 1 + 2^{-k} \cos x.$$

Therefore, $\lim_{k\to\infty} g_k(1) = 1$.

5. Suppose that f is C^2 and 2π-periodic. Suppose that $\int_0^{2\pi} |f'(x)|^2 = 1$. Prove that $|S_N f(x) - f(x)| \leq 10 N^{-1/2}$.

Since f is C^2 periodic, we know that $S_N f(x) \to f(x)$. Therefore,

$$|S_N f(x) - f(x)| \leq \sum_{|n|>N} |\hat{f}(n)|.$$

We let $g := f'$. On the other hand, by Parseval's theorem, we know that

$$1 \geq \|g\|^2 = \sum_n |\hat{g}(n)|^2 = \sum_{n=-\infty}^{\infty} |n|^2 |\hat{f}(n)|^2.$$

We will use this inequality and Cauchy-Schwarz to bound $\sum_{|n|>N} |\hat{f}(n)|$:

$$\sum_{|n|>N} |\hat{f}(n)| = \sum_{|n|>N} |\hat{f}(n)||n||n|^{-1} \leq \left(\sum_n |n|^2 |\hat{f}(n)|^2 \right)^{1/2} \left(\sum_{|n|>N} |n|^{-2} \right)^{1/2} \leq$$

$$\leq 1 \cdot (10N^{-1})^{1/2} \leq 10N^{-1/2}.$$