Practice Midterm

1. Suppose that f is continuous and 2π -periodic. Is it true that

$$\lim_{N \to \infty} \int_0^{2\pi} |S_N f(x) - f(x)| dx = 0?$$

Explain your answer.

2. Let g_N be the 2π -periodic function defined on $[-\pi, \pi]$ by setting $g_N(x) = \pi N$ if $|x| \leq 1/N$ and $g_N(x) = 1/N$ if $1/N \leq |x| \leq \pi$. Suppose that f is a C^0 and 2π -periodic function. Prove from first principles that $\lim_{N\to\infty} f * g_N(0) = f(0)$.

"Prove from first principles" means that you cannot cite the good kernel theorem, but you can imitate the proof of the good kernel theorem.

3. This problem is about the partial differential equation $\partial_t u(x,t) = \partial_x^3 u(x,t)$ (where $x \in \mathbb{R}$).

a.) Consider the initial value problem $\partial_t u(x,t) = \partial_x^3 u(x,t)$, with initial data u(x,0) = f(x), where f is a Schwartz function. Give a formula for the solution u(x,t) in terms of \hat{f} . You don't need to give a full proof that the formula holds.

b.) Using the formula from part a.), prove the following estimate. Suppose that

- $\int_{\mathbb{R}} |f(x)| dx \le 1$,
- $\int_{\mathbb{R}} |\partial_x^2 f(x)| dx \le 1.$

Prove that $|u(x,t)| \leq 100$ for all $x \in \mathbb{R}$ and all $t \geq 0$.

Hint: Use the information about f to control \hat{f} , and then use what you prove about \hat{f} to control u.

4. Suppose that $g(x) = 1 + \cos x$. In this problem, we consider what happens when we convolve g with itself many times. Here we use convolution in the setting of periodic functions, so

$$f_1 * f_2(x) := \frac{1}{2\pi} \int_0^{2\pi} f_1(y) f_2(x-y) dy.$$

Let $g_2 := g * g$. Let $g_{k+1} := g * g_k$. Find $\lim_{k \to \infty} g_k(1)$. Hint: Recall that if $f_3 = f_1 * f_2$, then $\hat{f}_3(n) = \hat{f}_1(n)\hat{f}_2(n)$.

5. Suppose that f is C^2 and 2π -periodic. Suppose that $\int_0^{2\pi} |f'(x)|^2 = 1$. Prove that $|S_N f(x) - f(x)| \le 10N^{-1/2}$.