Optional Open-Ended Project 1.

This is an open-ended project based on our discussion of probability and Fourier analysis.

Suppose that f is a Schwartz probability distribution: in other words, $f : \mathbb{R} \to \mathbb{R}$ is Schwartz and $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)dx = 1$.

We consider what happens when we convolve f with itself many times. Let $f_2 := f * f$ and $f_N = f * ... * f$, where we have N copies of f on the right-hand side. The goal of the exercise is to try to understand the behavior of f_N for large N.

For large N, f_N will look more and more like a Gaussian g_N. Using Fourier analysis, try to figure out which Gaussian is a good match for f_N in terms of basic information about f:

$$M_1(f) := \int_{\mathbb{R}} x f(x) dx.$$

$$M_2(f) := \int_{\mathbb{R}} x^2 f(x) dx.$$

(In general, the k^{th} moment of f is $M_k(f) := \int_{\mathbb{R}} x^k f(x)dx$.)

Try to find a formula for g_N in terms of $M_1(f)$ and $M_2(f)$. You might want to start with the special case that $M_1(f) = 0$.

Next it could be interesting to explore how close f_N is to g_N. There are a lot of different ways to measure how close f_N is to g_N. For example you could try to estimate $\int_{\mathbb{R}} |f_N(x) - g_N(x)|dx$. Note that f is normalized so that $\int_{\mathbb{R}} f(x)dx = 1$, and using this, you can show that $\int_{\mathbb{R}} f_N(x)dx = 1$ for all N. What information about f do you need to estimate $\int_{\mathbb{R}} |f_N(x) - g_N(x)|dx$? As a function of N, how rapidly does this integral go to zero?