Formulas from Math 103

Here is a list of formulas from our class. You can use this list during the midterm, and you can use any of these formulas on the exam.

Fourier series

In this section, f is a continuous 2π-periodic function.

Convolution: For 2π-periodic functions f and g, we normalize the convolution as

$$f * g(x) = (1/2\pi) \int_0^{2\pi} f(y) g(x-y) dy.$$

Convolution and Fourier series. If f, g are C^2 and 2π-periodic, then

$$(f * g)^\wedge(n) = \hat{f}(n)\hat{g}(n).$$

$$(f \cdot g)^\wedge(n) = \sum_{m=-\infty}^{\infty} \hat{f}(m)\hat{g}(n-m).$$

Partial Sums: We define $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$. Then $S_N f = f * D_N$, where D_N is the Dirichlet kernel defined as follows.

$$D_N(x) = \frac{\sin[(N+1/2)x]}{\sin[(1/2)x]}.$$

Fejer Partial Sums: We define the Fejer partial sum

$$\sigma_N f(x) = (1/N) \sum_{n=0}^{N-1} S_n f(x).$$

The Fejer partial sum can also be written in terms of the Fourier coefficients as

$$\sigma_N f(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}(n) e^{inx}.$$

The Fejer partial sum can also be written as a convolution $\sigma_N f = f * F_N$, where F_N is the Fejer kernel defined as follows:

$$F_N(x) = (1/N) \frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

The Fourier Transform

In this section f and g are Schwartz functions on $\mathbb{R}$.

The Fourier transform and translations

If $f_s(x) := f(x+s)$, then $\hat{f}_s(\omega) = e^{2\pi i \omega s} \hat{f}(\omega)$.

If $g(x) = e^{2\pi i \omega_0 x} f(x)$, then $\hat{g}(\omega) = \hat{f}(\omega - \omega_0)$.

The Fourier transform and derivatives:

If $g = \frac{\partial}{\partial x_i} f$, then $\hat{g}(\omega) = (2\pi i \omega_i) \hat{f}(\omega)$.

If $g(x) = (-2\pi i x_i) f(x)$, then $\hat{g}(\omega) = \frac{\partial}{\partial \xi_i} \hat{f}(\omega)$.

The Fourier transform and convolutions:

$$f * g(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

$$(f * g)^\wedge = \hat{f} \cdot \hat{g}.$$

$$(f \cdot g)^\wedge(\omega) = \hat{f} * \hat{g}.$$

The Fourier transform of a Gaussian:
If $f(x) = e^{-\pi|x|^2}$, then $\hat{f}(\omega) = e^{-\pi|\omega|^2}$.

Basic estimates

There are a couple basic estimates about sums and integrals that come up often and you don't need to justify them on the test.

$$\int_{\mathbb{R}} \frac{1}{1+x^2} dx \leq 4.$$

If $\alpha > 1$, and $N \geq 1$, then

$$\sum_{n=N}^{\infty} n^{-\alpha} \leq C(\alpha) N^{1-\alpha}.$$

As $\alpha \to 1$, the constant $C(\alpha) \to \infty$, but for $\alpha \geq 1.25$ you can take $C(\alpha) \leq 10$.