Solutions to Math 103 Final

1. Prove that there exists a Schwartz function $h : \mathbb{R} \to \mathbb{C}$ with the following property. If f_1 is any Schwartz function on $\mathbb{R}$ with $\operatorname{supp} \hat{f}_1 \subset [0,1]$ and f_2 is any Schwartz function on $\mathbb{R}$ with $\operatorname{supp} \hat{f}_2 \subset [2,3]$, then $(f_1 + f_2) * h = f_1$.

You don't need to write an exact formula for h. Just explain why the function h exists.

(This problem is related to how a radio works. Each radio station sends out a radio signal with frequency in a different range. The antennae of a radio receives a signal which is the sum of all of these contributions. To locate the signal from a single station, we need to find the part of the incoming signal in a given frequency range.)

Solution to 1: Let $g(\omega)$ be a C^∞ smooth function with $g(\omega) = 1$ for $\omega \in [0,1]$ and with the support of g contained in $[-1/2, 3/2]$. Let h be the inverse Fourier transform of g. Then $g = \hat{h}$. Since g is Schwartz, h is also Schwartz.

Now we consider $F := (f_1 + f_2) * h$. We consider the Fourier transform:

$$\hat{F} = (\hat{f}_1 + \hat{f}_2)\hat{h} = (\hat{f}_1 + \hat{f}_2)g.$$

Since $\hat{f}_1$ is supported in $[0,1]$, $\hat{f}_1 g = \hat{f}_1$. Since $\hat{f}_2$ is supported in $[2,3]$, $\hat{f}_2 g = 0$. Therefore, $\hat{F} = \hat{f}_1$. Since f_1 and F are Schwartz, we get by Fourier inversion that $F = f_1$. In other words, $(f_1 + f_2) * h = f_1$ as desired.

2. Suppose that f is a Schwartz function on $\mathbb{R}$. Suppose that $\int_{\mathbb{R}} |f|^2 = 1$ and that $\hat{f}$ is supported in $[-1,1]$. Prove that for any points $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le 1000|x - y|.$$

Solution to 2: First, by the fundamental theorem of calculus, we have

$$|f(x) - f(y)| = \left| \int_x^y f'(z) dz \right| \le |x - y| \max_z |f'(z)|.$$

Therefore, it suffices to prove that for all $z \in \mathbb{R}$, $|f'(z)| \le 1000$. Next we study f' using the Fourier transform. Let $g = f'$. Then $\hat{g} = 2\pi i \omega \hat{f}$. By Fourier inversion we get

$$f'(z) = \int_{\mathbb{R}} 2\pi i \omega \hat{f}(\omega) e^{2\pi i \omega z} d\omega.$$

Since $\hat{f}$ is supported in $[-1,1]$, we get

$$|f'(z)| \le 2\pi \int_{-1}^{1} |\omega||\hat{f}(\omega)|d\omega \le 2\pi \int_{-1}^{1} |\hat{f}(\omega)|d\omega.$$

Using Cauchy-Schwarz and then Plancherel we see that

$$\int_{-1}^{1} |\hat{f}(\omega)| \cdot 1 d\omega \le \left(\int_{-1}^{1} |\hat{f}(\omega)|^2 d\omega \right)^{1/2} \cdot (2)^{1/2} = 2^{1/2} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right) = 2^{1/2}.$$

So for every $z \in \mathbb{R}$, $|f'(z)| \le 2\pi \cdot 2^{1/2} \le 100$ as desired.

3. Suppose that $f \in L^1(\mathbb{R})$. Let

$$g = f * e^{-|x|^2} = \int_{\mathbb{R}} f(y) e^{-|x-y|^2} dy.$$

Prove that

$$\lim_{x \to +\infty} g(x) = 0.$$

Pick $\epsilon > 0$. We claim that there exists some $R < \infty$ so that $\int_{|y|>R} |f(y)|dy < \epsilon$. To prove the claim, we observe by the monotone (or Lebesgue dominated) convergence theorem that

$$\lim_{R \to \infty} \int_{-R}^{R} |f(y)|dy = \int_{\mathbb{R}} |f(y)|dy.$$

The integral $\int_{\mathbb{R}} |f(y)|dy$ is finite, and so

$$\int_{|y|>R} |f(y)|dy = \left(\int_{\mathbb{R}} |f(y)|dy \right) - \left(\int_{-R}^{R} |f(y)|dy \right) \to 0.$$

Now we estimate $g(x)$ for large x. In particular, for $x > R$, we see that

$$|g(x)| = \left| \int_{\mathbb{R}} f(y) e^{-|x-y|^2} dy \right| \le \left| \int_{|y| \le R} f(y) e^{-|x-y|^2} dy \right| + \left| \int_{|y|>R} f(y) e^{-|x-y|^2} dy \right|.$$

The second term on the right-hand side is bounded by $\int_{|y|>R} |f(y)|dy < \epsilon$. To bound the first term on the right-hand side, we note that since $|y| \le R$ and $x > R$, $e^{-|x-y|^2} \le e^{-(x-R)^2}$. Therefore, the first term is bounded by $e^{-(x-R)^2} \int_{\mathbb{R}} |f|$. All together, if $x > R$, we have

$$|g(x)| \le \epsilon + e^{-(x-R)^2} \|f\|_{L^1}.$$

If x is sufficiently large, we see that $|g(x)| < 2\epsilon$. Since ϵ is arbitrary, we see that $\lim_{x\to+\infty} g(x) = 0$.

4. Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) = 1$. Prove that there exists an open interval I with

$$m(E \cap I) \geq \frac{9}{10} m(I).$$

Solution to 4: Since $m(E)$ is finite, E can be well approximated by a finite union of intervals. More precisely, for any $\epsilon > 0$, there exists a finite union of intervals F so that $m(E \triangle F) < \epsilon$. Any finite union of intervals can be rewritten as a finite union of disjoint intervals in a unique way. So F is a finite disjoint union of intervals I_j. If $\epsilon < (1/100)$, then we claim that for one of these intervals I_j,

$$m(E \cap I_j) \geq \frac{9}{10} m(I_j).$$

Note that $m(F \cap E) \geq m(E) - m(F \triangle E) \geq 1 - \frac{1}{100} = \frac{99}{100}$. So

$$\sum_j m(E \cap I_j) = m(F \cap E) \geq \frac{99}{100}.$$

On the other hand, $m(F) \leq m(E) + m(F \triangle E) \leq 1 + \frac{1}{100} = \frac{101}{100}$. So

$$\sum_j m(I_j) \leq \frac{101}{100}.$$

Combining the last two equations, we see that

$$\sum_j m(E \cap I_j) \geq \frac{99}{101} \sum_j m(I_j).$$

Therefore, there must be some j so that

$$m(E \cap I_j) \geq \frac{99}{101} m(I_j) \geq \frac{9}{10} m(I_j).$$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^2 and 2π periodic. Suppose that

$$\frac{1}{2\pi} \int_0^{2\pi} f = 1$$

$$\max_x |f'(x)| \leq 9/10.$$

Let g_k be f convolved with itself k times. In other words, $g_1 := f$ and $g_k := g_{k-1} * f$. (Here we use convolution for 2π-periodic functions: $f * g := \frac{1}{2\pi} \int_0^{2\pi} f(y)g(x-y)dy$.)

Prove that g_{100} is strictly positive.

We study the Fourier series of f, and use it to study the Fourier series of g_k.

The first equation tells us that $\hat{f}(0) = 1$. We use the inequality $|f'(x)| \leq 9/10$ to bound the other Fourier coefficients. For $n \neq 0$, integrating by parts gives us

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx = -\frac{1}{2\pi} \cdot \frac{1}{-in} \int_0^{2\pi} f'(x)e^{-inx}dx.$$

Therefore,

$$|\hat{f}(n)| \leq |n|^{-1} \frac{1}{2\pi} \int_0^{2\pi} |f'(x)|dx \leq \frac{9}{10}|n|^{-1}.$$

Next we consider how $\hat{g}_k$ relates to $\hat{f}$. By definition $g_k = g_{k-1} * f$. Therefore, $\hat{g}_k = \hat{g}_{k-1}\hat{f}$. Since $g_1 = f$, clearly $\hat{g}_1 = \hat{f}$. Therefore, we see that $\hat{g}_k = (\hat{f})^k$. In particular, we see that

$$\hat{g}_k(0) = 1.$$

$$|\hat{g}_k(n)| \leq \left(\frac{9}{10}\right)^k |n|^{-k}.$$

If k is large, then $\hat{g}_k(n)$ becomes small for all $n \neq 0$. By Fourier inversion, we can write g_k in terms of its Fourier series as

$$g_k(x) = \sum_{n=-\infty}^{\infty} \hat{g}_k(n)e^{inx} = 1 + \sum_{n \neq 0, n \in \mathbb{Z}} \hat{g}_k(n)e^{inx}.$$

If k is large then the term 1 dominates the remaining term. In fact

$$\left| \sum_{n \neq 0, n \in \mathbb{Z}} \hat{g}_k(n)e^{inx} \right| \leq \sum_{n \neq 0, n \in \mathbb{Z}} |\hat{g}_k(n)| \leq 2 \cdot (9/10)^k \sum_{n=1}^{\infty} |n|^{-k}.$$

If $k = 100$, then it follows easily that this last expression is at most $1/2$. Therefore, we get

$$|g_{100}(x) - 1| \leq 1/2.$$

Since g_{100} is real, we see that $g_{100}(x) > 0$ for all x.

6. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function with $\int_{\mathbb{R}} |f|^2 = 1$ and with $\operatorname{supp} \hat{f} \subset [1,2]$. Suppose that $u(x,t)$ solves the Schrodinger equation $\partial_t u = i\partial_x^2 u$ with initial conditions $u(x,0) = f(x)$, (and that u is Schwartz uniform).

Suppose in addition that

$$|f(x)| \leq 100(1+|x|)^{-3}. \tag{1}$$

Prove that for all $t > 0$,

$$|u(0,t)| < 10^{100} t^{-1}. \tag{$*$}$$

Physical interpretation: The solution to the Schrodinger equation models a quantum mechanical particle. The integral $\int_a^b |u(x,t)|^2 dx$ gives the probability that the particle lies in the interval $[a,b]$ at time t. Therefore, $|u(x,t)|^2$ can be interpreted as the 'probability density' that the particle is at the point x at time t. The condition that $\operatorname{supp} \hat{f} \subset [1,2]$ says that the particle has "momentum" between 1 and 2. Equation 1 implies that at time 0, the particle lies fairly close to 0 with high probability. Since the particle has momentum in the range $[1,2]$, physical intuition suggests that it is unlikely to be near zero when t is large.

Solution to 6: We consider the Fourier transform of u. As we learned in class, $\hat{u}(\omega,t)$ obeys the differential equation

$$\partial_t \hat{u}(\omega,t) = -4\pi^2 i\omega^2 \hat{u}(\omega,t).$$

We have the initial condition $\hat{u}(\omega,0) = \hat{f}(\omega)$, and therefore

$$\hat{u}(\omega,t) = e^{-4\pi^2 i\omega^2 t} \hat{f}(\omega).$$

By Fourier inversion, we see that

$$u(0,t) = \int_{\mathbb{R}} e^{-4\pi^2 i\omega^2 t} \hat{f}(\omega) d\omega.$$

Next we want to use the hypotheses about f to control $\hat{f}$. First of all, since $\hat{f}$ is supported in $[1,2]$, we can write

$$u(0,t) = \int_1^2 e^{-4\pi^2 i\omega^2 t} \hat{f}(\omega) d\omega. \tag{1}$$

The bound $|f(x)| \leq 100(1+|x|)^{-3}$ allows us to bound both $\hat{f}(\omega)$ and its derivative. First we bound $|\hat{f}(\omega)|$.

$$|\hat{f}(\omega)| \leq \int_{\mathbb{R}} |f(x)| dx \leq 100 \int_{\mathbb{R}} (1+|x|)^{-3} dx \leq 1000.$$

To bound the derivative of $\hat{f}(\omega)$, we first recall that

$$\frac{d}{d\omega}\hat{f}(\omega) = \int_{\mathbb{R}} (-2\pi i x) f(x) e^{-2\pi i \omega x} dx.$$

Using the estimate $|f(x)| \leq 100(1+|x|)^{-3}$, we get the bound

$$\left| \frac{d}{d\omega}\hat{f}(\omega) \right| \leq 100 \int_{\mathbb{R}} (2\pi|x|)(1+|x|)^{-3} dx \leq 10^4.$$

Now we use these estimates for $\hat{f}$ and its derivative to control the integral in (1). We want to prove that the oscillation in the factor $e^{-4\pi^2 i\omega^2 t}$, together with the regularity of $\hat{f}$, leads to cancellation in the integral. Because the oscillatory term has the form $e^{-4\pi^2 i\omega^2 t}$ we change variables to $\eta = \omega^2$. We have $d\eta = 2\omega d\omega$, and so $d\omega = (1/2)\eta^{-1/2} d\eta$. Therefore, the integral (1) becomes

$$u(0,t) = \frac{1}{2} \int_1^4 e^{-4\pi^2 i\eta t} \hat{f}(\eta^{1/2}) \eta^{-1/2} d\eta.$$

We abbreviate $g(\eta) = \hat{f}(\eta^{1/2})\eta^{-1/2}$. Then we get

$$u(0,t) = \frac{1}{2} \int_1^4 g(\eta) e^{-4\pi^2 i\eta t} d\eta.$$

We will control this integral by integrating by parts. We integrate by parts with $u = g(\eta)$ and $dv = e^{-4\pi^2 i\eta t} d\eta$. We note that g is a C^1 function on $[1,4]$, and that g vanishes at the endpoints of $[1,4]$. Because of this vanishing, the boundary terms vanish when we integrate by parts, and we get

$$u(0,t) = \frac{1}{2} \int_1^4 g(\eta) e^{-4\pi^2 i\eta t} d\eta = -\frac{1}{-8\pi^2 i t} \int_1^4 g'(\eta) e^{-4\pi^2 i\eta t} d\eta.$$

Therefore, we get

$$|u(0,t)| \leq t^{-1} \max_{\eta \in [1,4]} |g'(\eta)|.$$

It just remains to bound $|g'(\eta)|$. Using the Liebniz rule and the chain rule, we see that

$$g'(\eta) = \left(\hat{f}(\eta^{1/2})\eta^{-1/2} \right)' = \hat{f}'(\eta^{1/2}) \cdot \left(\frac{1}{2}\eta^{-1/2} \right) \eta^{-1/2} + \hat{f}(\eta^{1/2}) \left(\frac{-1}{2}\eta^{-3/2} \right).$$

When $\eta \in [1,4]$, negative powers of η are at most 1. Using our bounds $|\hat{f}(\omega)| \leq 10^3$ and $|\hat{f}'(\omega)| \leq 10^4$, we see that

$$\max_{\eta \in [1,4]} |g'(\eta)| \leq 10^4 + 10^3 \leq 10^5.$$

Therefore, we get all together

$$|u(0,t)| \leq 10^5 t^{-1}.$$

Final remarks: If f decays faster, we can prove even better decay for $|u(0,t)|$. Given a bound of the form

$$|f(x)| \leq C(1+|x|)^{-m},$$

we can bound $|\hat{f}(\omega)|$ and we can bound the derivatives $|\frac{d^k}{d\omega^k}\hat{f}(\omega)|$ for $1 \leq k \leq m-2$. Following the same strategy and integrating by parts $m-2$ times, we can prove the following stronger bound for $|u(0,t)|$:

$$|u(0,t)| \leq C' t^{-(m-2)}.$$