

Math 103 Final

Instructions: This is an in-class final exam. It is closed book, but you can use the formula sheet provided, which recalls some basic formulas from Fourier analysis and PDE.

If you can't solve a problem completely, write down what you tried and say where you got stuck. If you have some intuition that you can't make completely rigorous, write that down too. Good luck!

1. Prove that there exists a Schwartz function $h : \mathbb{R} \rightarrow \mathbb{C}$ with the following property. If f_1 is any Schwartz function on \mathbb{R} with $\text{supp } \hat{f}_1 \subset [0, 1]$ and f_2 is any Schwartz function on \mathbb{R} with $\text{supp } \hat{f}_2 \subset [2, 3]$, then $(f_1 + f_2) * h = f_1$.

You don't need to write an exact formula for h . Just explain why the function h exists.

(This problem is related to how a radio works. Each radio station sends out a radio signal with frequency in a different range. The antennae of a radio receives a signal which is the sum of all of these contributions. To locate the signal from a single station, we need to find the part of the incoming signal in a given frequency range.)

2. Suppose that f is a Schwartz function on \mathbb{R} . Suppose that $\int_{\mathbb{R}} |f|^2 = 1$ and that \hat{f} is supported in $[-1, 1]$. Prove that for any points $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq 1000|x - y|.$$

3. Suppose that $f \in L^1(\mathbb{R})$. Let

$$g = f * e^{-|x|^2} = \int_{\mathbb{R}} f(y)e^{-|x-y|^2} dy.$$

Prove that

$$\lim_{x \rightarrow +\infty} g(x) = 0.$$

4. Suppose that $E \subset \mathbb{R}$ is a measurable set with $m(E) = 1$. Prove that there exists an open interval I with

$$m(E \cap I) \geq \frac{9}{10}m(I).$$

5. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^2 and 2π periodic. Suppose that

$$\frac{1}{2\pi} \int_0^{2\pi} f = 1$$

$$\max_x |f'(x)| \leq 9/10.$$

Let g_k be f convolved with itself k times. In other words, $g_1 := f$ and $g_k := g_{k-1} * f$. (Here we use convolution for 2π -periodic functions: $f * g := \frac{1}{2\pi} \int_0^{2\pi} f(y)g(x-y)dy$.)

Prove that g_{100} is strictly positive.

6. Suppose that $f : \mathbb{R} \rightarrow \mathbb{C}$ is a Schwartz function with $\int_{\mathbb{R}} |f|^2 = 1$ and with $\text{supp } \hat{f} \subset [1, 2]$. Suppose that $u(x, t)$ solves the Schrodinger equation $\partial_t u = i\partial_x^2 u$ with initial conditions $u(x, 0) = f(x)$, (and that u is Schwartz uniform).

Suppose in addition that

$$(1) \quad |f(x)| \leq 100(1 + |x|)^{-3}.$$

Prove that for all $t > 0$,

$$|u(0, t)| < 10^{100}t^{-1}. \quad (*)$$

Physical interpretation: The solution to the Schrodinger equation models a quantum mechanical particle. The integral $\int_a^b |u(x, t)|^2 dx$ gives the probability that the particle lies in the interval $[a, b]$ at time t . Therefore, $|u(x, t)|^2$ can be interpreted as the ‘probability density’ that the particle is at the point x at time t . The condition that $\text{supp } \hat{f} \subset [1, 2]$ says that the particle has “momentum” between 1 and 2. Equation ?? implies that at time 0, the particle lies fairly close to 0 with high probability. Since the particle has momentum in the range $[1, 2]$, physical intuition suggests that it is unlikely to be near zero when t is large.