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Metaphors in systolic geometry

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Abstract. We discuss the systolic inequality for n-dimensional tori, explaining different metaphors that help to organize the proof. The metaphors connect systolic geometry with minimal surface theory, topological dimension theory, and scalar curvature.

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1. Introduction

This essay is an introduction to systolic geometry. Rather than surveying a lot of results, I'm going to focus on one central result, and I want to survey a lot of ways of thinking about it.

Systolic inequality for tori. *(Gromov, 1983 [10]) If (T^n, g) is an n-dimensional torus with a Riemannian metric, then there is a non-contractible curve $\gamma \subset T^n$ whose length obeys the inequality*

$$length(\gamma) \le C_n Vol(T^n, g)^{1/n}.$$

This inequality is very general. It holds in every dimension n, and it holds for every metric g on T^n. (For example, there is no restriction on the curvature of g.) Because it applies to so many metrics, the result is remarkable.

In the early 80's, Gromov formulated several remarkable metaphors connecting the systolic inequality to important ideas in other areas of geometry, and these metaphors have guided most of the research in the subject. They connect the systolic problem with ideas about minimal surfaces, topological dimension, and scalar curvature. The main goal of this essay is to explain Gromov's metaphors.

The systole of (T^n, g) is defined to be the length of the shortest non-contractible curve in (T^n, g). We will denote it by $Sys(T^n, g)$. The systole of (T^n, g) and the volume of (T^n, g) are both ways of describing the *size* of (T^n, g). Size may sound like a basic issue in Riemannian geometry, but mathematicians have not spent much time exploring it. The proofs of the systolic inequality lead to some interesting perspectives about size in Riemannian geometry. At the end of the essay, I will discuss the issue of size and point out some open problems.

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2. Examples

To get a feeling for the systolic inequality, let's consider some examples.

First, suppose that (T^n, g) is a product of circles with lengths $L_1, ..., L_n$. The length of the shortest non-contractible curve in this metric is $\min_{i=1}^n L_i$, and the volume of the metric is $\prod_{i=1}^n L_i$. Hence we see that for product metrics, there is a non-contractible curve of length at most $Vol^{1/n}$.

Next let's consider some examples of two dimensional tori that we can visualize. The systolic inequality for two-dimensional tori was proven by Loewner in 1949 with a sharp constant.

Loewner's systolic inequality. *(1949) If (T^2, g) is a 2-dimensional torus with a Riemannian metric, then there is a non-contractible curve $\gamma \subset (T^2, g)$ whose length obeys the inequality*

$$length(\gamma) \leq C Area(T^2, g)^{1/2},$$

where $C = 2^{1/2} 3^{-1/4} \sim 1.1$.

The diagram below shows four different tori.

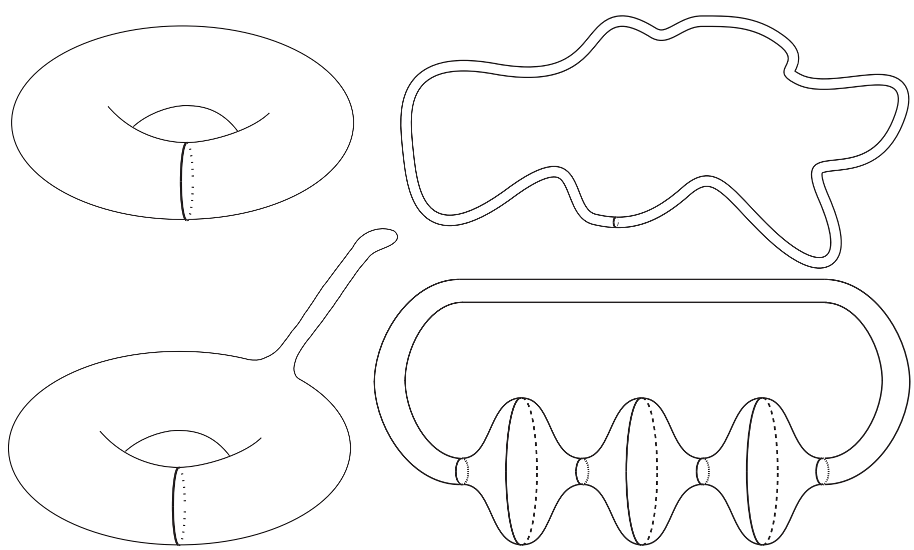

Figure 1. Pictures of tori

The first picture is supposed to show a torus of revolution, where we take the circle of radius 1 around the point $(2, 0)$ is the x-z plane and revolve it around the z-axis. It has systole 2π and area around 60, and so it obeys the systolic inequality. According to Loewner's theorem, there is nothing we can do to dramatically increase the systole while keeping the area the same. The second picture shows a long skinny torus. When we make the torus skinnier and longer, the systole goes down and the area stays about the same. The third picture shows a torus with a long thin spike coming out of it. When we add a long thin spike to the torus, the systole doesn't change and the spike adds to the area. The fourth picture shows a ridged torus with some thick parts and some thin parts. When we put ridges in the surface of the torus, the systole only depends on the thinnest part and the thick parts contribute heavily to the area.

(Friendly challenge to the reader: can you think of a torus with geometry radically different from the pictures above?)

These pictures start to give a feel for the systolic inequality in two dimensions. In three dimensions it gets much harder to draw pictures. In fact, in three dimensions, there are examples of metrics much stranger than these. We touch on them more in the next section.

3. Why is the systolic inequality hard?

The systolic inequality has the same flavor as the isoperimetric inequality. To get a sense of the difficulty of the systolic inequality, let's recall the classical isoperimetric inequality and then compare them.

Isoperimetric inequality. *Suppose that $U \subset \mathbb{R}^n$ is a bounded open set. Then the volume of the boundary ∂U and the volume of U are related by the formula*

$$Vol_n(U) \leq C_n Vol_{n-1}(\partial U)^{\frac{n}{n-1}}.$$

From the Riemannian point of view, this domain U is a compact manifold with boundary equipped with a *flat* Riemannian metric (the Euclidean metric). The isoperimetric inequality can be considered as a theorem about flat Riemannian metrics. By contrast, the systolic inequality is a theorem about *all* Riemannian metrics on T^n. (To make the comparison tighter, the classical isoperimetric inequality holds for every *flat* metric on the n-ball. The systolic inequality does not make sense on a ball, but we will meet below a covering inequality that holds for *every* metric on the n-ball.) Now the set of flat metrics is only a tiny sliver in the set of all metrics. Moreover, the flat metrics are probably the easiest metrics to understand. So we see that the systolic inequality is far more general than the classical isoperimetric inequality.

Loewner proved the systolic inequality for two-dimensional tori in 1949, but the three-dimensional case was open for more than thirty years after Loewner's proof. Why is three dimensions so much harder than two? The space of Riemannian metrics has many strange examples, disproving naive conjectures, and this is especially true in dimensions three and higher. For example, let us consider the following problem, raised by Berger and Gromov. Suppose that g is a metric on $S^n \times S^n$ with volume 1. Can we find a non-trivial copy of S^n with controlled n-dimensional volume? When $n = 1$, this is the systolic inequality for T^2. By analogy, it seems plausible that it should hold for all n, but it turns out that there are counterexamples for $n \geq 2$.

Gromov-Katz examples. *([28]) For each $n \geq 2$, and every number B, there is a metric on $S^n \times S^n$ with (2n-dimensional) volume 1, so that every non-contractible n-sphere in $S^n \times S^n$ has (n-dimensional) volume at least B.*

As we go from domains in Euclidean space to metrics on T^2 to metrics on T^3, the possible geometries become more complicated. To get a perspective on this, let me describe a naive conjecture about the sizes of level sets and trace how it plays out in the different settings.

Naive conjecture 1. *If $U \subset \mathbb{R}^n$ is a bounded open set, then there is a function $f : U \to \mathbb{R}$ so that the volume of every level set is controlled by the volume of U:*

$$\text{For every } y \in \mathbb{R},\ Vol_{n-1}[f^{-1}(y)] \leq C_n Vol_n(U)^{\frac{n-1}{n}}.$$

Naive conjecture 1 is true. I proved it in [18].

Naive conjecture 2. *If g is a metric on T^2, then there is a function $f : T^2 \to \mathbb{R}$ so that the length of every level set is controlled by the area of g:*

$$\text{For every } y \in \mathbb{R},\ Length[f^{-1}(y)] \leq C Area(T^2, g)^{1/2}.$$

Naive conjecture 2 is also true. This result is more surprising than the first one. The problem was open for a long time. It was proven by Balacheff and Sabourau in [5].

Naive conjecture 3. *If g is a metric on T^3, then there is a function $f : T^3 \to \mathbb{R}$ so that the area of every level set is controlled by the volume of (T^3, g):*

$$\text{For every } y \in \mathbb{R},\ Area[f^{-1}(y)] \leq C Vol(T^3, g)^{2/3}.$$

Naive conjecture 3 is wrong. (There are many counterexamples. I think that historically the first examples came from work of Brooks.)

This story is typical for naive conjectures in metric geometry. The space of all the metrics on T^3 is huge. There is a substantial zoo of strange examples, and there are probably many other strange metrics yet to be discovered. Universal statements about all metrics on T^3 are rare and significant.

4. The role of metaphors in systolic geometry

Reminiscing about his work in systolic geometry, Gromov wrote, "Since the setting was so plain and transparent, I expected rather straightforward proofs." (See the end of Chapter 4 in [11] for Gromov's recollections of working on the systolic problem.) But in spite of the plain and transparent setting, the result is difficult, and in particular, it's hard to see how to get started. In the early 1980's, he formulated several remarkable metaphors connecting the systolic inequality to important ideas in other areas of geometry. Guided by these metaphors, he proved the systolic inequality. We now have three independent proofs of the systolic inequality for the n-dimensional torus, each based on a different metaphor.

The goal of this essay is to explain Gromov's metaphors. In doing that, I hope to describe the flavor of this branch of geometry and put it into a broad context. The metaphors connect the systolic inequality to the following areas:

1. Minimal surface theory.
2. Topological dimension theory.
3. Scalar curvature.

Each metaphor gives a valuable perspective about the systolic problem and suggests an outline of the proof. It still takes substantial work to fill in the details of the proofs. Up to the present, every proof of the systolic inequality is based on one of these metaphors.

5. Minimal surface theory

In the early 1970's, Bombieri and Simon [6] proved the following sharp inequality about the geometry of minimal surfaces in Euclidean space.

Bombieri-Simon radius inequality. *Suppose that Z^n is a closed submanifold of $\mathbb{R}^N$, and that Y^{n+1} is a minimal surface with $\partial Y = Z$. Suppose that Z has the same volume as a round n-sphere of radius R. Then for each point $y \in Y$, the distance from y to Z is at most R.*

This inequality is sharp when Z is a round sphere of radius R and Y is the corresponding ball of radius R.

Using this inequality, Bombieri and Simon proved the Gehring link conjecture. If Z^n and W^{N-n-1} are disjoint closed surfaces in $\mathbb{R}^N$, then the linking number of Z with W is defined as follows. Let Y^{n+1} be a surface with $\partial Y = Z$. Put Y in general position, and consider $Y \cap W$, which will be a finite set of points. If we count these points with multiplicity we get the linking number of Z with W. This linking number doesn't depend on the choice of Y. If the number is non-zero, we say that Z and W are linked.

Gehring link conjecture. *Suppose that Z^n and W^{N-n-1} are linked submanifolds of $\mathbb{R}^N$. If Z has the same volume as a round n-sphere of radius R, then the distance from Z to W is at most R. In other words, there are points $z \in Z$ and $w \in W$ with $|z - w| \leq R$.*

Proof. By the solution of the Plateau problem, there is a minimal surface Y with $\partial Y = Z$. Since Z and W are linked, Y must intersect W in some point $y \in W$. But by the radius inequality, the distance from y to Z is at most R. □

Gromov built an analogy between the Gehring link conjecture and the systolic problem. On the one hand, such an analogy sounds promising because both inequalities bound a 1-dimensional length (or distance) in terms of an n-dimensional volume.

$$Dist(Z^n, W^{N-n-1}) \leq C_n Vol(Z)^{1/n}. \qquad \text{(Gehring link inequality)}$$

$$Sys(T^n, g) \leq C_n Vol(T^n, g)^{1/n}. \qquad \text{(Systolic inequality)}$$

On the other hand, the analogy sounds far-fetched because the systolic problem is about an abstract Riemannian manifold, and the Gehring link conjecture is about a submanifold of Euclidean space $\mathbb{R}^N$.

Every closed Riemannian manifold admits a canonical embedding into a Banach space.

Kuratowski embedding. *Define the map $K : (M^n, g) \to L^\infty(M)$ by letting $K(p)$ be the distance function $dist_p$. The map K is an isometry in the strong sense that*

$$dist_{(M,g)}(p, q) = \|K(p) - K(q)\|_{L^\infty}.$$

The Kuratowski embedding is canonical and respects the geometry of (M, g). The target space $L^\infty(M)$ is infinite-dimensional, but we can approximate this embedding using a finite-dimensional Banach space. For each (M, g) there is a finite dimension N and an embedding $K_0 : (M, g) \to (\mathbb{R}^N, l^\infty)$ which is nearly isometric in the sense that

$$\frac{99}{100}\|K_0(p) - K_0(q)\|_{l^\infty} \leq dist_{(M,g)}(p, q) \leq \frac{100}{99}\|K_0(p) - K_0(q)\|_{l^\infty}.$$

The following striking observation relates the systole problem and the linking problem.

Linking observation. *([**?**])0 Let (T^n, g) be any Riemannian metric on T^n. Let Z^n be the image $K_0(T^n) \subset (\mathbb{R}^N, l^\infty)$. Then Z is linked with a surface W^{N-n-1} with $dist(Z, W) \geq (1/8)Sys(T^n, g)$.*

We know that Z is linked with a faraway surface W, and we wish to conclude that Z has a large volume. This is a version of the Gehring link problem in $(\mathbb{R}^N, l^\infty)$.

Metaphor 1. *The systolic inequality is like the Gehring link problem in the Banach space $(\mathbb{R}^N, l^\infty)$.*

The method of Bombieri-Simon does not work in Banach spaces. In effect, their method uses the symmetry of Euclidean space. To get estimates for linked surfaces in $(\mathbb{R}^N, l^\infty)$, Gromov proved the following inequality.

Filling radius inequality. *([**?**]) If $Z^n \subset (\mathbb{R}^N, l^\infty)$ is a closed surface, then there exists a surface Y^{n+1} with $\partial Y = Z$ such that for each $y \in Y$,*

$$dist(y, Z) \leq C_n Vol_n(Z)^{1/n}.$$

The filling radius inequality implies a linking inequality in $(\mathbb{R}^N, l^\infty)$: if Z^n and W^{N-n-1} are linked in $(\mathbb{R}^N, l^\infty)$, then $dist(Z, W) \leq C_n Vol(Z)^{1/n}$. To prove the systolic inequality, we let $Z = K_0(T^n, g)$ and we let W be the surface mentioned in the linking observation above. Then we observe that

$$(1/8)Sys(T^n, g) \leq dist(Z, W) \leq C_n Vol(Z)^{1/n} \sim C_n Vol(T^n, g)^{1/n}.$$

There is an important story about the constant C_n in Gromov's filling radius inequality. It's comparatively easy to prove an inequality of the form $dist(y, Z) \leq C_N Vol_n(Z)^{1/n}$ with a constant C_N depending on the ambient dimension N. This inequality does not imply the systolic inequality. We can find a nearly isometric embedding from (T^n, g) into some $(\mathbb{R}^N, l^\infty)$, but the dimension N depends on the metric g. Roughly speaking, if g is complicated, then N will be large. To prove the systolic inequality for all g, we need a filling radius estimate for all N with a uniform constant. We discuss this issue more in Section 8 below.

(A note on vocabulary: I've been using the word surface a little bit loosely. For readers with background in geometric measure theory, surface means Lipschitz chain and closed surface means Lipschitz cycle. For readers with less background, surfaces (or Lipschitz chains) include smooth submanifolds and they are a little bit more general. A surface is a submanifold with mild singularities. For example, suppose that Z is a submanifold diffeomorphic to $\mathbb{CP}^2$. By the cobordism theory, $\mathbb{CP}^2$ is not the boundary of any 5-dimensional manifold. In this case, Y may be homeomorphic to a cone over $\mathbb{CP}^2$, which is a manifold except for one singularity at the cone point.)

6. Topological dimension theory

In the 1870's, Cantor discovered that $\mathbb{R}^q$ and $\mathbb{R}^n$ have the same cardinality even if $q < n$. This discovery surprised and disturbed him. He and Dedekind formulated the question whether $\mathbb{R}^q$ and $\mathbb{R}^n$ are homeomorphic for $q < n$. This question turned out to be quite difficult. It was settled by Brouwer in 1909.

Topological Invariance of Dimension. *(Brouwer 1909) If $q < n$, then there is no homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^q$.*

Cantor and Dedekind certainly knew that $\mathbb{R}^q$ and $\mathbb{R}^n$ were not *linearly* isomorphic. Linear algebra gives us two stronger statements:

Linear algebra lemma 1. *If $q < n$, then there is no surjective linear map from $\mathbb{R}^q$ to $\mathbb{R}^n$.*

Linear algebra lemma 2. *If $q < n$, then there is no injective linear map from $\mathbb{R}^n$ to $\mathbb{R}^q$.*

It seems reasonable to try to prove topological invariance of dimension by generalizing these lemmas. A priori, it's not clear which lemma is more promising. Cantor spent a long time trying to generalize Lemma 1 to continuous maps. (At one point, Cantor even believed he had succeeded [27].) In fact, Lemma 1 does not generalize to continuous maps.

Space-filling curve. *(Peano, 1890) For any $q < n$, there is a surjective continuous map from $\mathbb{R}^q$ to $\mathbb{R}^n$.*

In his important paper on topological invariance of dimension, Brouwer proved that Lemma 2 does generalize to continuous maps.

Brouwer non-embedding theorem. *If $n > q$, then there is no injective continuous map from $\mathbb{R}^n$ to $\mathbb{R}^q$.*

So it turns out that Lemma 2 is more robust than Lemma 1. A smaller-dimensional space may be stretched to cover a higher-dimensional space. But a higher-dimensional space may not be squeezed to fit into a lower-dimensional space. This fact is not obvious a priori - it is an important piece of acquired wisdom in topology. In this section, we're going to talk about the geometric consequences/cousins of this fundamental discovery of topology.

Shortly after Brouwer, Lebesgue introduced a nice approach to Brouwer's non-embedding theorem in terms of coverings. If U_i is an open cover of some set $X \subset \mathbb{R}^n$, we say that the multiplicity of the cover is at most μ if each point $x \in X$ is contained in at most μ open sets U_i. We say the diameter of a cover is at most D if each open set U_i has diameter at most D. For any $\epsilon > 0$, Lebesgue constructed an open cover of $\mathbb{R}^n$ with multiplicity $\leq n+1$ and diameter at most ϵ. He then proposed the following lemma.

Lebesgue covering lemma. *If U_i are open sets that cover the unit n-cube, and each U_i has diameter less than 1, then some point of the n-cube lies in at least $n+1$ different U_i.*

(Brouwer gave the first proof of the Lebesgue covering lemma in 1913. See the interesting essay "The emergence of topological dimension theory" [27] for more information on the history.)

To see how the Lebesgue covering lemma implies the non-embedding theorem, suppose that we have a continuous map f from the unit n-cube to $\mathbb{R}^q$ for some $q < n$. Lebesgue constructed an open cover U_i of $\mathbb{R}^q$ with multiplicity $q+1$ and diameter $< \epsilon$. The preimages $f^{-1}(U_i)$ form an open cover of the unit n-cube with multiplicity $q+1$. Since $q+1 < n+1$, the Lebesgue covering lemma implies that some set $f^{-1}(U_i)$ must have diameter at least 1. On the other hand, the diameters of the sets U_i are as small as we like. By taking a limit as $\epsilon \to 0$, we can find a point $y \in \mathbb{R}^q$ such that the fiber $f^{-1}(y)$ has diameter at least 1. So the Lebesgue covering lemma implies the following large fiber lemma:

Large fiber lemma. *Suppose $q < n$. If f is a continuous map from the unit n-cube to $\mathbb{R}^q$, then one of the fibers of f has diameter at least 1. In other words, there exist points p, q in the unit n-cube with $|p - q| \geq 1$ and $f(p) = f(q)$.*

The large fiber lemma is a precise quantitative theorem saying that an n-dimensional cube cannot be squeezed into a lower-dimensional space.

What is it about the unit n-cube which makes it hard to cover with multiplicity n? Roughly speaking, the key point is that the unit n-cube is "fairly big in all directions". If every non-contractible curve in (T^n, g) has length at least 1, then in some sense, (T^n, g) is fairly big in all directions too. Gromov was able to make this precise and proved the following generalization of the Lebesgue covering lemma.

Generalized Lebesgue covering lemma. *([?]) Suppose that g is a Riemannian metric on the n-dimensional torus T^n with systole at least 1. In other words, every non-contractible loop in (M^n, g) has length at least 1.*

If U_i is an open cover of (M^n, g) with diameter at most 1/10, then some point of M lies in at least n+1 different sets U_i.

Topologists following Lebesgue used the covering lemma as a basis for defining the dimension of metric spaces [26]. They said that the Lebesgue covering dimension of a metric space X is at most n if X admits open covers with multiplicity at most $n + 1$ and arbitrarily small diameters. Different notions of dimension were intensively studied in the first half of the twentieth century. The most well-known is the Hausdorff dimension of a metric space. The Hausdorff dimension and the Lebesgue covering dimension may be different. For example, the Cantor set has Lebesgue dimension zero and Hausdorff dimension strictly greater than zero. In 1937, Szpilrajn proved that $LebDim(X) \leq HausDim(X)$ for any compact metric space X. To do so, he constructed coverings of metric spaces with small diameters and bounded multiplicities.

Szpilrajn covering construction. *(1937) If X is a (compact) metric space with n-dimensional Hausdorff measure 0, and $\epsilon > 0$ is any number, then there is a covering of X with multiplicity at most n and diameter at most ϵ. Hence X has Lebesgue dimension $\leq n - 1$.*

Gromov asked whether Szpilrajn's theorem is stable in the following sense: If X has very small n-dimensional Hausdorff measure, is there a covering of X with multiplicity at most n and small diameter? In 2008, I constructed such coverings for Riemannian manifolds.

Covering construction for Riemannian manifolds. *(Guth 2008, [19]) If (M^n, g) is an n-dimensional Riemannian manifold with volume V, then there is an open cover of (M^n, g) with multiplicity n and diameter at most $C_n V^{1/n}$.*

Combining this covering construction with the generalized Lebesgue covering lemma, we get a second proof of the systolic inequality. The second proof is summarized in the following metaphor.

Metaphor 2. *The systolic inequality is like topological dimension theory. In particular, it follows from robust versions of the Lebesgue covering lemma and the Szpilrajn covering construction.*

The inequality in my covering construction above and Gromov's filling radius inequality are actually quite similar to each other. The covering inequality implies the filling radius inequality, but the results are equally useful in practice. The methods of proof are quite different though. The proof of the covering construction uses ideas from topological dimension theory: we begin by choosing an open cover of (M, g) and mapping to the nerve of the cover. The main difficulty is that we need quantitative estimates that don't appear in topological dimension theory. We need to estimate the multiplicity the cover, the sizes of the open sets and their overlaps, etc. Taking classical ideas from topology and modifying them to get quantitative estimates is a developing area of research connecting geometry and topology. See Gromov's essay 'Quantitative topology' [15] for an introduction.

7. Scalar curvature

The Geroch conjecture was one of the guiding problems in the history of scalar curvature.

Geroch conjecture. *The n-torus does not admit a metric of positive scalar curvature.*

In the late 1970's, there were two breakthroughs in the field of scalar curvature. Schoen and Yau invented the minimal hypersurface method, and used it to prove the Geroch conjecture for $n \leq 7$ (see [33] and [34]). We will discuss the minimal hypersurface method more below. Shortly afterwards, Gromov and Lawson used the Dirac operator method to prove the Geroch conjecture for all n.

Gromov's third metaphor connects the Geroch conjecture to the systolic inequality. The metaphor is based on the description of scalar curvature in terms of the volumes of small balls.

Scalar curvature and volumes of balls. *If (M^n, g) is a Riemannian manifold and p is a point in M, then the volumes of small balls in M obey the following asymptotic:*

$$VolB(p, r) = \omega_n r^n - c_n Sc(p) r^{n+2} + O(r^{n+3}). \qquad (*)$$

In this equation, ω_n is the volume of the unit n-ball in Euclidean space, and $c_n > 0$ is a dimensional constant. So we see that if $Sc(p) > 0$, then the volumes of tiny balls $B(p, r)$ are a bit less than Euclidean, and if $Sc(p) < 0$ then the volumes of tiny balls are a bit more than Euclidean.

The scalar curvature measures the asymptotic behavior of volumes of tiny balls as the radius goes to zero. We will consider something analogous to scalar curvature but based on the volumes of balls with finite radius - we call it the "macroscopic scalar curvature at scale r". We define the macroscopic scalar curvature as follows. Let p be a point in (M^n, g). We let $V(p, r)$ be the volume of the ball of radius r around p. Then we let $\tilde{V}(p, r)$ be the volume of the ball of radius r around p in the universal cover of M. (We'll come back in a minute to discuss why it makes sense to use the universal cover here.) Now we compare the volume $\tilde{V}(p, r)$ with the volumes of balls of radius r in spaces of constant curvature. We let $\tilde{V}_S(r)$ denote the volume of the ball of radius r in a simply connected space with constant curvature and scalar curvature S. If we fix r, then $\tilde{V}_S(r)$ is a decreasing function of S; as $S \to +\infty$, $\tilde{V}_S(r)$ goes to zero, and as $S \to -\infty$, $\tilde{V}_S(r)$ goes to infinity. We define the "macroscopic scalar curvature at scale r at p" to be the number S so that $\tilde{V}(p,r) = \tilde{V}_S(r)$.

We denote the macroscopic scalar curvature at scale r at p by $Sc_r(p)$. In particular, if $\tilde{V}(p,r)$ is more than $\omega_n r^n$, then $Sc_r(p) < 0$, and if $\tilde{V}(p,r) < \omega_n r^n$, then $Sc_r(p) > 0$.

By formula $(*)$, it's straightforward to check that $\lim_{r\to 0} Sc_r(p) = Sc(p)$.

Let's work out a simple example. Suppose that g is a flat metric on the n-dimensional torus T^n. In this case, the universal cover of (T^n, g) is Euclidean space. Therefore, we have $\tilde{V}(p,r) = \omega_n r^n$ for each $p \in T^n$ and each $r > 0$. Hence $Sc_r(p) = 0$ for every r and p. If we had used volumes of balls in (T^n, g) instead of in the universal cover, then we would have $Sc_r(p) > 0$ for all r bigger than the diameter of (T^n, g). By using the universal cover, we arrange that flat metrics have $Sc_r = 0$ at every scale r.

Metaphor 3. *The macroscopic scalar curvature is like the scalar curvature.*

This metaphor leads to some deep, elementary, and wide open conjectures in Riemannian geometry.

Generalized Geroch conjecture. *(Gromov 1985) Fix $r > 0$. The n-dimensional torus does not admit a metric with $Sc_r > 0$. Equivalently, if g is any metric on T^n, then the universal cover (T^n, g) contains a ball of radius r and volume at least $\omega_n r^n$.*

The generalized Geroch conjecture is very powerful (if it's true). Since the scalar curvature is the limit of Sc_r as $r \to 0$, the generalized Geroch conjecture implies the original Geroch conjecture. The generalized Geroch conjecture also implies the systolic inequality, which we can see as follows. Suppose that (T^n, g) has systole at least 1. The generalized Geroch conjecture implies that the universal cover of (T^n, g) contains a ball of radius $(1/2)$ and volume $\geq \omega_n (1/2)^n$. Since the systole of (T^n, g) is at least 1, the covering projection $\tilde{T}^n \to T^n$ is injective on this ball. Therefore, (T^n, g) contains a ball of radius $(1/2)$ and volume at least $\omega_n (1/2)^n$. In particular, the total volume of (T^n, g) must be at least $\omega_n (1/2)^n$.

The generalized Geroch conjecture really appeals to me because it's so strong and so elementary to state, but I don't see any plausible tool for approaching the problem.

Now we return to the Schoen-Yau proof of the Geroch conjecture, and we discuss how to adapt it to systolic geometry. The key idea in the Schoen-Yau proof is an inequality for stable minimal hypersurfaces in a manifold of positive scalar curvature.

Stability inequality for scalar curvature. *If (M^n, g) is a Riemannian manifold with $Sc > 0$, and $\Sigma^{n-1} \subset M$ is a stable minimal hypersurface, then Σ has - on average - positive scalar curvature also.*

To see how to apply this observation, suppose that (M^3, g) has positive scalar curvature. Then a stable minimal hypersurface $\Sigma \subset M^3$ is 2-dimensional, and it has (on average) positive scalar curvature. In two dimensions, the scalar curvature is much better understood, and it's not so hard to get topological and geometric information about Σ. Now we know topological and geometric information about every minimal surface Σ in M, and we can use this to learn topological and geometric information about M itself. With this tool, Schoen and Yau proved the Geroch conjecture.

I proved an analogue of the Schoen-Yau stability inequality using volumes of balls instead of scalar curvature. Informally, the lemma says that if a Riemannian manifold has balls of small volume then an absolutely minimizing hypersurface also has balls of small volume.

Stability inequality for volumes of balls. *(Guth, 2009, [20]) Suppose that (M^n, g) is a Riemannian manifold where every ball of radius 1 has volume at most α, and suppose that (M, g) has systole at least 2. If $\Sigma^{n-1} \subset M$ is an embedded surface which is absolutely minimizing in its homology class, then every ball in Σ of radius $1/2$ has (n-1)-volume at most 2α.*

Using this lemma, I proved a weak version of the generalized Geroch conjecture with a non-sharp constant.

Non-sharp generalized Geroch. *(Guth, 2009, [20]) For any metric g on T^n, the universal cover of T^n contains a ball of radius 1 and volume at least $c(n) > 0$. Therefore, if (T^n, g) has systole at least 2, then it contains a ball of radius 1 with volume at least $c(n) > 0$.*

It's unknown whether there is any systolic analogue of the Dirac operator method for positive scalar curvature.

The results of Schoen-Yau and Gromov-Lawson remain today the main theorems about scalar curvature. Now we turn to an open question in the field of scalar curvature, and we consider it from the viewpoint of systolic geometry.

Schoen conjecture. *Suppose that (M^n, hyp) is a closed hyperbolic manifold. Suppose that g is any metric on M obeying the scalar curvature estimate $Sc(g) \geq Sc(hyp)$. Then $Vol(M, g) \geq Vol(M, hyp)$.*

This elegant conjecture appears in connection with the Yamabe problem in conformal geometry [32], and it is also beautiful in its own right. In two dimensions, the conjecture follows from the Gauss-Bonnet formula. In three dimensions, it was proven by Perelman as a byproduct of the Ricci flow proof of geometrization. In four dimensions, the conjecture is open, but LeBrun proved a cousin of this conjecture for complex hyperbolic manifolds [31]. LeBrun's proof uses Seiberg-Witten theory. In dimensions $n \geq 5$, the problem is wide open. According to a deep theorem of Besson, Courtois, and Gallot, if $Ric(g) \geq Ric(hyp)$, then $Vol(M, g) \geq Vol(M, hyp)$ [4]. This theorem of Besson, Courtois, and Gallot is much weaker than the Schoen conjecture, but it is still a landmark result in comparison geometry. In dimensions $n \geq 5$ we don't have any lower bound at all for $Vol(M^n, g)$ with $Scal(g) \geq Scal(hyp)$.

The Schoen conjecture can be generalized to the macroscopic scalar curvature, producing an even more general and daunting conjecture.

Generalized Schoen conjecture. *Let $r > 0$ be any number. Suppose that (M^n, hyp) is a closed hyperbolic manifold. Suppose that g is any metric on M obeying the estimate $Sc_r(g) \geq Sc_r(hyp)$. Then $Vol(M, g) \geq Vol(M, hyp)$.*

Needless to say, this conjecture is far out of reach. But using methods from systolic geometry, I proved a weak version of this conjecture with a non-sharp constant.

Non-sharp generalized Schoen conjecture. *(Guth, [22]) Suppose that (M^n, hyp) is a hyperbolic manifold. Suppose that g is any metric on M obeying the estimate $Sc_1(g) \geq Sc_1(hyp)$. In other words, every unit ball in the universal cover of (M^n, g) has volume at most the volume of a hyperbolic unit ball. Then $Vol(M, g) \geq c(n) Vol(M, hyp)$.*

The generalized Schoen conjecture implies the original Schoen conjecture by taking the limit as $r \to 0$, but my inequality is not sharp enough to give any information about scalar curvature.

The minimal hypersurface approach to scalar curvature is not enough to resolve the Schoen conjecture. Similarly, the minimal hypersurface approach to systolic geometry is not enough to prove the volume estimate above. The proof of this volume estimate uses the techniques coming from topological dimension theory.

8. The Federer-Fleming averaging argument

The three metaphors we have been discussing provide large-scale perspective on the systolic problem. They provide guidance about how the outline of the proof should go, but they usually don't provide guidance about how the details of the proof should go. One crucial idea that makes the details work is the Federer-Fleming averaging argument. It is the one ingredient which appears in some form in all three proofs of the systolic inequality.

Here is the first example of the Federer-Fleming averaging argument, coming from their 1959 paper [9] on the Plateau problem.

Deformation lemma. *Suppose that z^k is a k-dimensional surface in the unit N-ball B^N, and that z has a boundary ∂z lying in ∂B^N. If $k < N$, then there is a map $\Phi : z \to \partial B^N$ which fixes ∂z and obeys the volume estimate*

$$Vol_k[\Phi(z)] \leq C(k, N) Vol_k[z].$$

Informally, the proposition says that we can push z into the boundary of the ball without stretching it too much.

The simplest way one could think to map z into ∂B^N is to project z radially outward to the boundary. Let Φ_0 denote the radial projection outward from zero. In polar coordinates, $\Phi_0(r, \theta) = (1, \theta)$. This map Φ_0 is undefined at the point 0, but we can first put z into general position so that it avoids 0, and this operation has a negligible effect on the volume of z. But the radial projection Φ_0 may not obey the volume estimate. If a large fraction of z is concentrated near to 0, then the radial projection may badly stretch this portion of z leading to an image with a huge volume. Instead of projecting from 0, one can instead project outward from any point $p \in B^N$. We let $\Phi_p : B^N \setminus \{p\} \to \partial B^N$ denote the radial projection outward from the point p. Federer and Fleming discovered that for any fixed surface z, *most projections Φ_p obey the volume estimate*. To do that, they estimated the average volume of a projection, proving the inequality

$$\frac{1}{Vol B^N} \int_{B^N} Vol_k[\Phi_p z] dp \leq C(k, N) Vol_k z.$$

This inequality follows in a couple lines using Fubini's theorem.

This simple averaging method tells us something fundamental about surface areas. By using the averaging method many times, one can prove a surprising range of geometric estimates about surface areas. This approach to geometry problems originates with Federer and Fleming in 1959, but Gromov's proof of the systolic inequality really showed how powerful it is, starting a stream of results proven by using the averaging trick many times. Let's trace the history of this method.

1. (Isoperimetric inequalities) The method begins with Federer and Fleming who used the deformation lemma to prove a general isoperimetric inequality [9].

   Federer-Fleming isoperimetric inequality. *If Z is a k-dimensional closed surface in $\mathbb{R}^N$, then there is a (k+1)-dimensional surface Y with $\partial Y = Z$ obeying the volume estimate*

   $$Vol_{k+1}(Y) \leq C(k, N) Vol_k(Z)^{\frac{k+1}{k}}.$$

   Their proof also gives a filling radius estimate.

   Federer-Fleming filling radius inequality. *If Z is a k-dimensional closed surface in $\mathbb{R}^N$, then there is a (k+1)-dimensional surface Y with $\partial Y = Z$ so that every point $y \in Y$ obeys the distance estimate*

   $$dist(y, Z) \leq C(k, N) Vol_k(Z)^{\frac{1}{k}}.$$

2. (Isoperimetric inequalities in high dimensions) The constants in the Federer-Fleming estimates above are not sharp. They are particularly bad in large ambient dimensions N. As $N \to \infty$, the constant $c(k, N) \to \infty$. The sharp constants were found using geometric measure theory, and they occur when Z is a round sphere. (The sharp radius estimate is due to Bombieri-Simon [6] and the sharp isoperimetric inequality is due to Almgren [1].) In particular, the sharp constants do not depend on the ambient dimension N.

   Let us contrast the Federer-Fleming approach with the minimal surface approach. In the minimal surface approach to the filling radius inequality, one takes Y to be an absolutely minimizing chain with boundary Z. The existence of such a minimizer is a deep theorem (the solution of the Plateau problem). The variational method really doesn't tell us how to construct Y or even how to approximate Y. Next one proves that Y is smooth at most points. Finally, minimal surfaces enjoy special geometric properties such as the monotonicity formula, which then imply estimates about the radius or volume of Y. By contrast, Federer and Fleming construct the filling Y "by hand", using the deformation lemma repeatedly. This construction is crude compared to the minimal surface filling, and hence it does not give sharp constants.

In the early 80's, one might have guessed that a direct construction of Y would be too crude to prove good isoperimetric estimates when the ambient dimension $N \to \infty$. Surprisingly, Gromov was able to adapt the Federer-Fleming method to prove isoperimetric and filling radius estimates with constants independent of the ambient dimension [11]. Moreover, the method was flexible enough to work in Banach spaces such as $(\mathbb{R}^N, l^\infty)$, where minimal surface techniques do not work. The main new idea in Gromov's proof was to use induction on k. The proof was further simplified and generalized by Wenger in [35]. His proof is only a couple pages long.

Isoperimetric inequality in Banach spaces. *Let B be a Banach space. Suppose that Z is a k-dimensional closed surface in B. Then there is a (k+1)-dimensional surface Y with $\partial Y = Z$ obeying the volume inequaliy*

$$Vol_{k+1}(Y) \leq C(k) Vol_k(Z)^{\frac{k+1}{k}}.$$

3. (Sweep out inequalities) In an appendix to [11], Gromov used the Federer-Fleming method to approach the Almgren sweepout inequality.

   Sweep out inequality. *(Almgren, 1962 [2]) Suppose that $\Phi : S^k \times S^{n-k} \to S^n$ is a map of non-zero degree. Equip the target S^n with the standard unit sphere metric. Then there exists some $\theta \in S^{n-k}$ so that $\Phi(S^k \times \{\theta\})$ has k-volume at least the volume of the unit k-sphere.*

   This is a deep result based on the variational theory of minimal surfaces. For a reader without a strong background in geometric measure theory, the proof is hundreds of pages long. Gromov proved a slightly weaker result by using the Federer-Fleming averaging lemma repeatedly. The lower bound on volume in Gromov's result is a non-sharp constant $c(k, n) > 0$, but the proof is only a few pages long.

4. (Isoperimetric inequalities on Lie groups) Gromov adapted the Federer-Fleming method to Lie groups such as the Heisenberg group. In [16] he proved an analogue of the filling radius inequality for surfaces in the Heisenberg group.

   Building on Gromov's work, Young proved an isoperimetric inequality in the Heisenberg group as follows.

Isoperimetric inequality in the Heisenberg group. *(Young, 2008, [36]) Let (H^{2n+1}, g) be a left-invariant metric on the Heisenberg group H^{2n+1}. If Z is a k-dimensional closed surface in H^{2n+1} and $k < n$, then there is a (k+1)-dimensional surface Y with $\partial Y = Z$ obeying the volume estimate*

$$Vol_{k+1}(Y) \le C(k,n,g) Vol_k(Z)^{\frac{k+1}{k}}.$$

Young's main new idea was to use the averaging lemma at many scales.

5. (Area-expanding embeddings) I applied the Federer-Fleming method to the problem of area-expanding embeddings. If $U, V \subset \mathbb{R}^n$ are open sets, an embedding $\Psi : U \to V$ is called k-expanding if it increases the k-dimensional area of each k-dimensional surface. I studied when there is a k-expanding embedding from one n-dimensional rectangle into another, and I answered the question up to a constant factor [23]. This problem turns out to be fairly "rigid" in the sense that the optimal strategy for embedding one rectangle in another is simple. The difficult part of the problem is to prove that there are no k-expanding embeddings between certain rectangles.

   Area-expanding embeddings of rectangles. *If R is an n-dimensional rectangle with side lengths $R_1 \le ... \le R_n$, and R' is an n-dimensional rectangle with side lengths $R'_1 \le ... \le R'_n$, and if there is a k-expanding embedding from R into R', then the following inequalities hold*

   $$R_1 ... R_j (R_{j+1} ... R_l)^{\frac{k-j}{l-j}} \le C(n) R'_1 ... R'_j (R'_{j+1} ... R'_l)^{\frac{k-j}{l-j}},$$

   for each $1 \le j \le k$ and $k \le l \le n$.

   Up to a constant factor, this list of inequalities is necessary and sufficient to find a k-expanding from R into R'.

6. (Point selection theorem in combinatorics) Gromov applied the Federer-Fleming method to give a new proof of the point selection theorem in combinatorics.

   Point selection. *(Barany [3]) If $p_1, ..., p_N$ are points in $\mathbb{R}^n$, consider the $\binom{N}{n+1}$ n-dimensional simplices with vertices among these points. Then there is a point $y \in \mathbb{R}^n$ which lies in at least $c(n)\binom{N}{n+1}$ of the $\binom{N}{n+1}$ n-simplices, for a universal constant $c(n) \ge (n+1)^{-(n+1)}$.*

   Gromov reproved this theorem and generalized it. Given N points in $\mathbb{R}^n$, we get a linear map L from the (N-1)-simplex Δ^{N-1} to $\mathbb{R}^n$, given by mapping the N vertices of the simplex to $p_1, ..., p_N$. The point selection theorem says that y lies in the image of at least $c(n)\binom{N}{n+1}$ of the n-faces of Δ^{N-1}. It turns out that this holds for all continuous maps, not only for linear maps.

Topological simplex inequality. *(Gromov, 2009, [14]) Suppose that F is a continuous map from Δ^{N-1} to $\mathbb{R}^n$. Then there is a point $y \in \mathbb{R}^n$ which lies in the image of at least $c(n)\binom{N}{n+1}$ n-faces of Δ^{N-1}.*

Gromov's proof of this combinatorial theorem is closely based on his proof of the sweepout inequality, using a combinatorial analogue of the Federer-Fleming averaging argument.

In each of these theorems, using the Federer-Fleming averaging trick over and over is essentially the entire proof.

I want to end this section with a philosophical discussion of the Federer-Fleming averaging method.

The fundamental idea is that the average value of some function may be easier to understand than the function itself. This idea is certainly older than Federer and Fleming. As a dramatic example, Erdos used a similar averaging trick to prove that there are colorings of a graph with no cliques. Given appropriate bounds on the size of the graph and the size of the cliques, he proved that the average number of cliques in a coloring is less than 1. Hence colorings with no cliques exist, even though it is difficult to produce an explicit example. Federer and Fleming borrowed this idea and used it to prove inequalities in geometry. (It would be interesting to know more about the history of this averaging trick.)

The wonderful thing about the averaging trick is that it's so flexible. As we have seen, some of the results in the above list can also be approached by minimal surface theory, and the minimal surface techniques lead to the sharp constants. Using the averaging lemma repeatedly is not as precise but it's more flexible. It can be adapted to Banach spaces. It can be adapted to the Heisenberg group. It can be adapted to the geometry of surfaces inside a rectangle - measuring how the dimensions of the rectangle influence the isoperimetric inequalities. It can be adapted to the combinatorics of an N-dimensional simplex with $N \to \infty$.

In the small field of metric geometry, the Federer-Fleming averaging trick is the most common tool. When the averaging trick doesn't work, we often get stuck. Intuitively, we can only use the averaging trick to find a geometric object if the objects we are looking for are pretty common. Are there any geometric theorems about the existence of rare objects? What tools could we use to find those objects?

I think these issues may be related to the open problems at the end of this essay. Those problems have to do with notions of size in Riemannian geometry, and I need to lay a little groundwork before we get to them.

9. Notions of size in Riemannian geometry

Many of the arguments in systolic geometry have to do with various ways of measuring the 'size' of a Riemannian manifold.

Size invariants. *Let M be a smooth manifold. A size invariant for metrics on M is a function S which assigns a positive number to each metric on M, and which obeys the following axioms.*

1. If g and g' are isometric, then $S(g) = S(g')$.
2. If $g \leq g'$, then $S(g) \leq S(g')$.

(We say that $g \leq g'$ if for each point x and each tangent vector v in $T_x M$, $g(v,v) \leq g'(v,v)$.)

The volume and diameter are two fundamental size invariants. Many Riemannian invariants are not size invariants. For example, anything related to the curvature is not a size invariant. The injectivity radius is not a size invariant, and neither are the eigenvalues of the Laplacian or the lengths of closed geodesics. But the systole is a size invariant.

The most interesting size invariants I know came out of the proofs of the systolic inequality. We met these invariants implicitly in the discussion above, and now we turn our attention to them.

Filling radius. *If (M^n, g) is a closed Riemannian manifold, then we define its filling radius to be the smallest radius R so that the Kuratowski embedding of (M, g) into L^∞ bounds a chain inside its R-neighborhood.*

Uryson width. *If X is any metric space, such as a Riemannian manifold, and $q \geq 0$ is an integer, then we say that X has q-dimensional Uryson width at most W if there is an open cover of X with diameter $\leq W$ and multiplicity $\leq q+1$. We denote the q-dimensional Uryson width of X by $UW_q(X)$.*

Among the size invariants that I know, the Uryson width seems like the most useful one, so I will try to give a little intuition about it. In some sense, the definition goes back to topologists working on dimension theory, including Uryson. Gromov returned to the the definition and applied it to Riemannian geometry. He gives a long discussion of it in [17]. Recall that $\mathbb{R}^n$ has open covers of multiplicity $n{+}1$ with arbitrarily small diameters, so $UW_n(\mathbb{R}^n, g_{euclid}) = 0$. More generally, the Uryson n-width of any n-dimensional simplicial complex is equal to zero. Roughly speaking, X has a small Uryson q-width if it "looks q-dimensional". If X has an open cover with multiplicity $q+1$, then the nerve of the cover is a simplicial complex of dimension q. There is a continuous map Φ from X to the nerve so that each fiber of the map is contained in one of the open sets. Thus a metric space X with small q-dimensional Uryson width may be mapped into a q-dimensional complex and each fiber of the map will have small diameter. If the Uryson q-width of X is $< \epsilon$, then we can informally say, "when we look at X from far away and cannot distinguish points of distance $< \epsilon$, X appears to be q-dimensional".

So far in this essay, we have seen three universal inequalities about size functions.

1. The systolic inequality: $Sys(g) \leq C(n) Vol(g)^{1/n}$ for all metrics on T^n.

2. The filling radius inequality: $FillRad(g) \leq C(n) Vol(g)^{1/n}$ for all metrics on closed n-manifolds.

3. The Uryson width inequality: $UW_{n-1}(g) \leq C(n) Vol(g)^{1/n}$ for all metrics on n-manifolds.

These inequalities are closely related. The Uryson width inequality implies the filling radius inequality which implies the systolic inequality, but they all come from the same circle of ideas. Twenty-five years ago, Gromov proved 1 and 2 and conjectured 3. Since then, we have not found any really new universal inequality about sizes of Riemannian metrics. The inequalities we have proven since are either much easier than the filling radius inequality or else they are closely related to the filling radius inequality.

Are there other interesting universal inequalities about the sizes of Riemannian manifolds?

There may well be, but let me try to describe why it hasn't been easy to find any. It is easy to define size invariants of Riemannian manifolds. I know ten or twenty different kinds of size invariants for Riemannian manifolds. But it's often hard to evaluate these invariants, even roughly. For example, here is a simple size invariant for metrics on S^3.

Covering radius. *The covering radius of (S^3, g) is the smallest radius R so that we can find a degree 1 contracting map from the 3-sphere of radius R to (S^3, g).*

(A contracting map is a map that decreases distances.) The manifold S^3 is diffeomorphic to the Lie group $SU(2)$. The left-invariant metrics on $SU(2)$ are some of the simplest metrics on S^3. Gromov raised the problem of estimating the covering radius of left-invariant metrics on $SU(2)$. There is a huge gap between the best known upper and lower bounds, and the problem has been open for more than twenty five years.

There are lots of size invariants, and they are often hard to evaluate. I don't know any good perspective to organize the information. As we've seen, the space of Riemannian metrics is huge, so there are counterexamples for many naive conjectures about size invariants. And after defining ten or twenty size invariants it gets hard to see what's significant.

I want to end by putting forward two questions about sizes of Riemannian manifolds. I think that whether the answers are yes or no, some interesting new geometry will be involved.

The first question is about the geometry of high-genus surfaces. My main point is that we really don't have a good understanding of the geometry of high-genus Riemannian surfaces.

Question 1. *(Buser) If (Σ^2, g) is a closed Riemannian surface of arbitrary genus, is there a continuous map F from Σ to a graph Γ obeying the following inequality:*

$$\text{for every } y \text{ in } \Gamma,\ Length[F^{-1}(y)] \leq C Area(\Sigma, g)^{1/2}?$$

(This question is a small variation on Buser's question about the sharp value of the Bers constant — see [7].)

This question connects to topics we've seen above in a couple ways. First of all, the Uryson width inequality tells us that we can find a map F from (Σ, g) to a graph so that each fiber has *diameter* at most $C Area(\Sigma, g)^{1/2}$. This estimate does not imply the length estimate at all, because a fiber may be a very long curve which wiggles a lot and therefore has a small diameter. The most interesting examples of high genus Riemannian surfaces are probably the arithmetic hyperbolic surfaces studied by Buser and Sarnak in [8]. These surfaces have genus G, area around G, and diameter around $\log G$. Since the entire surface has diameter around $\log G$, any curve in it has diameter at most around $\log G$. When G is large, the diameters are much smaller than the square root of the area. So any map from an arithmetic hyperbolic surface to a graph has fibers of diameter at most $Area^{1/2}$, but it's not at all clear how small we can make the *lengths* of the fibers.

This question also fits in with the naive conjectures in Section 3 of this essay. In particular, if Σ is a small genus surface, then Balacheff and Sabourau proved that the answer to the question is yes. In a bit more generality, here is their result.

Balacheff-Sabourau inequality. *([5]) If (Σ^2, g) is a closed surface of genus G, then there is a function $f : \Sigma^2 \to \mathbb{R}$ so that for every $y \in \mathbb{R}$, the length of the level set $f^{-1}(y)$ obeys the inequality*

$$Length[f^{-1}(y)] \le C\sqrt{G+1}Area(\Sigma^2, g)^{1/2}.$$

For large genus, the right-hand side grows like $\sqrt{G}$, and this behavior is sharp. But if we allow maps to a 1-dimensional complex Γ instead of maps to $\mathbb{R}$, we may get a better estimate for lengths. If the answer to Question 1 is yes, then we can look for similar inequalities in higher dimensions. Can every 3-manifold of volume 1 be mapped to a 2-dimensional complex with fibers of length $\le C$? Can every 3-manifold of volume 1 be mapped to $\mathbb{R}^2$ with fibers of length $\le C$?? Can every 3-manifold of volume 1 be mapped to a 1-dimensional complex with fibers of area $\le C$?

The second problem is about Uryson widths. Recall the Uryson width inequality, $UW_{n-1}(M^n, g) \le C(n)Vol(M^n, g)^{1/n}$, which says that an n-manifold of tiny n-dimensional volume looks (n-1)-dimensional. What conditions on g would force (M^n, g) to look (n-2)-dimensional?

This is an open-ended question that could go in many directions. For instance, Gromov has a conjecture that if the scalar curvature of g is at least 1, then $UW_{n-2}(M^n, g) \le C(n)$.

Here is another direction suggested by the geometry of area-contracting maps. Suppose that M^n is just the standard unit n-ball, and we have the metric g_{ij} written in coordinates. What do we need to know pointwise about g_{ij} to control $UW_{n-2}(B^n, g)$?

Question 2. *Let B^n denote the standard (unit) n-ball in $\mathbb{R}^n$, and let g_0 denote the standard Euclidean metric. Suppose that g is another metric obeying $\Lambda^k g \le \Lambda^k g_0$. This means that for every k-dimensional surface $\Sigma^k \subset B^n$, the g-volume of Σ is at most the Euclidean volume of Σ. Suppose that $n/k \ge d$. Is it true that $UW_{n-d}(B^n, g) \le C(n)$?*

To get a sense of this question, let us first imagine that the metric $g_{ij}(x)$ is constant in x. In this case, (B^n, g_{ij}) is isometric to a Euclidean ellipsoid. If g is a constant metric and $\Lambda^k g \le \Lambda^k g_0$, then linear algebra implies that $UW_{k-1}(B^n, g) \le 1$. At this point, one might naively conjecture that all metrics g with $\Lambda^k g \le \Lambda^k g_0$ obey $UW_{k-1}(B^n, g) \le C(n)$. Moreover, the Uryson width inequality implies that if $\Lambda^n g \leq \Lambda^n g_0$, then $UW_{n-1}(B^n, g) \leq C(n)$. So the naive conjecture is true when $k = n$. But the naive conjecture is false for other values of k because of a counterexample coming from work of Zel'dovitch in astrophysics and Gehring in conformal geometry. Zel'dovitch's work has to do with the internal geometry of a neutron star. I think that this counterexample is the worst case, and the question asks whether this is true. See my paper [24] on area-contracting maps and topology for more context.

10. Reading guide

For the reader who would like to learn more about this area of geometry, here are some resources.

Gromov wrote about systolic geometry in several places. The key research paper is "Filling Riemannian manifolds" [10]. His expository writing about systoles includes Chapter 4 of *Metric Structures* [11], and the essay "Systoles and isosystolic inqualities" [13].

Katz's expository work on systoles includes the book *Systolic Geometry and Topology* [29] and his website on systoles [30]. The website contains a lot of interesting stuff, including a list of open problems in the field.

I wrote a set of notes on the systolic inequality [21] which explains the original proof in detail in 14 pages. This talk is based on my essay [25], which includes several topics we didn't have time to discuss here: hyperbolic geometry, symmetry, calibrations, and Nabutovsky's work on the complexity of the space of metrics.

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