

PARALLEL CHIP-FIRING ON THE COMPLETE GRAPH: DEVIL'S STAIRCASE AND POINCARÉ ROTATION NUMBER

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ABSTRACT. We study how parallel chip-firing on the complete graph K_n changes behavior as we vary the total number of chips. Surprisingly, the activity of the system, defined as the average number of firings per time step, does not increase smoothly in the number of chips; instead it remains constant over long intervals, punctuated by sudden jumps. In the large n limit we find a “devil’s staircase” dependence of activity on the number of chips. The proof proceeds by reducing the chip-firing dynamics to iteration of a self-map of the circle S^1 , in such a way that the activity of the chip-firing state equals the Poincaré rotation number of the circle map. The stairs of the devil’s staircase correspond to periodic chip-firing states of small period.

1. INTRODUCTION

In this paper we explore a connection between the Poincaré rotation number of a circle map $S^1 \rightarrow S^1$ and the behavior of a discrete dynamical system known variously as *parallel chip-firing* [4, 5] or the *fixed energy sandpile* [2, 14]. We use this connection to shed light on two intriguing features of parallel chip-firing, *mode locking* and *short period attractors*. Ever since Bagnoli, Ceconi, Flammini, and Vespignani [1] found evidence of mode locking and short period attractors in numerical experiments in 2003, these two phenomena have called out for a mathematical explanation. This paper is intended as a first step toward such an explanation.

In parallel chip-firing on the complete graph K_n , each vertex $v \in [n] = \{1, \dots, n\}$ starts with a pile of $\sigma(v) \geq 0$ chips. A vertex with n or more chips is *unstable*, and can *fire* by sending one chip to each vertex of K_n (including one chip to itself). The *parallel update* rule fires all unstable vertices simultaneously, yielding a new chip configuration $U\sigma$ given by

$$(1) \quad U\sigma(v) = \begin{cases} \sigma(v) + r(\sigma), & \sigma(v) \leq n - 1 \\ \sigma(v) - n + r(\sigma), & \sigma(v) \geq n. \end{cases}$$

Date: November 17, 2008.

2000 Mathematics Subject Classification. 26A30, 37E10, 37E45, 82C20.

Key words and phrases. Circle map, devil’s staircase, fixed-energy sandpile, mode locking, non-ergodicity, parallel chip-firing, rotation number, short period attractors.

The author is supported by a National Science Foundation Postdoctoral Research Fellowship.

Here

$$r(\sigma) = \#\{v \mid \sigma(v) \geq n\}$$

is the number of unstable vertices. Write $U^0\sigma = \sigma$, and $U^t\sigma = U(U^{t-1}\sigma)$ for $t \geq 1$.

Note that the total number of chips in the system is conserved. In particular, only finitely many different states are reachable from the initial configuration σ , so the sequence $(U^t\sigma)_{t \geq 0}$ is eventually periodic: there exist integers $m \geq 1$ and $t_0 \geq 0$ such that

$$(2) \quad U^{t+m}\sigma = U^t\sigma \quad \forall t \geq t_0.$$

The *activity* of σ is the limit

$$(3) \quad a(\sigma) = \lim_{t \rightarrow \infty} \frac{\alpha_t}{nt}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings performed in the first t updates. By (2), the limit in (3) exists and equals $\frac{1}{mn}(\alpha_{t+m} - \alpha_t)$ for any $t \geq t_0$. Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

Following [1], we ask: how does the activity change when chips are added to the system? If σ_n is a chip configuration on K_n , write $\sigma_n + k$ for the configuration obtained from σ_n by adding k chips at each vertex. The function

$$\tilde{s}_n(k) = a(\sigma_n + k)$$

is called the *activity phase diagram* of σ_n . Surprisingly, for many choices of σ_n , the function \tilde{s}_n looks like a staircase, with long intervals of constancy punctuated by sudden jumps (Figure 1). This phenomenon is known as *mode locking*: if the system is in a preferred mode, corresponding to a wide stair in the staircase, then even a relatively large perturbation in the form of adding extra chips will not change the activity. For a general discussion of mode locking, see [11].

To quantify the idea of mode locking in our setting, suppose we are given an infinite family of chip configurations $\sigma_2, \sigma_3, \dots$ with σ_n defined on K_n . Suppose that $0 \leq \sigma_n(v) \leq n-1$ for all $v \in [n]$, and that for all $0 \leq x \leq 1$

$$(4) \quad \frac{1}{n} \#\{v \in [n] \mid \sigma_n(v) < nx\} \rightarrow F(x) \quad \text{as } n \rightarrow \infty$$

for a continuous function $F : [0, 1] \rightarrow [0, 1]$. Then according to Theorem 8, the activity phase diagrams \tilde{s}_n , suitably rescaled, converge pointwise to a continuous, nondecreasing function $s : [0, 1] \rightarrow [0, 1]$. Moreover, under mild additional hypotheses, Proposition 10 says that this limiting function s is a *devil's staircase*: it is locally constant on an open dense subset of $[0, 1]$. For each rational number $p/q \in [0, 1]$ there is a stair of height p/q , that is, an interval of positive length on which s is constant and equal to p/q .

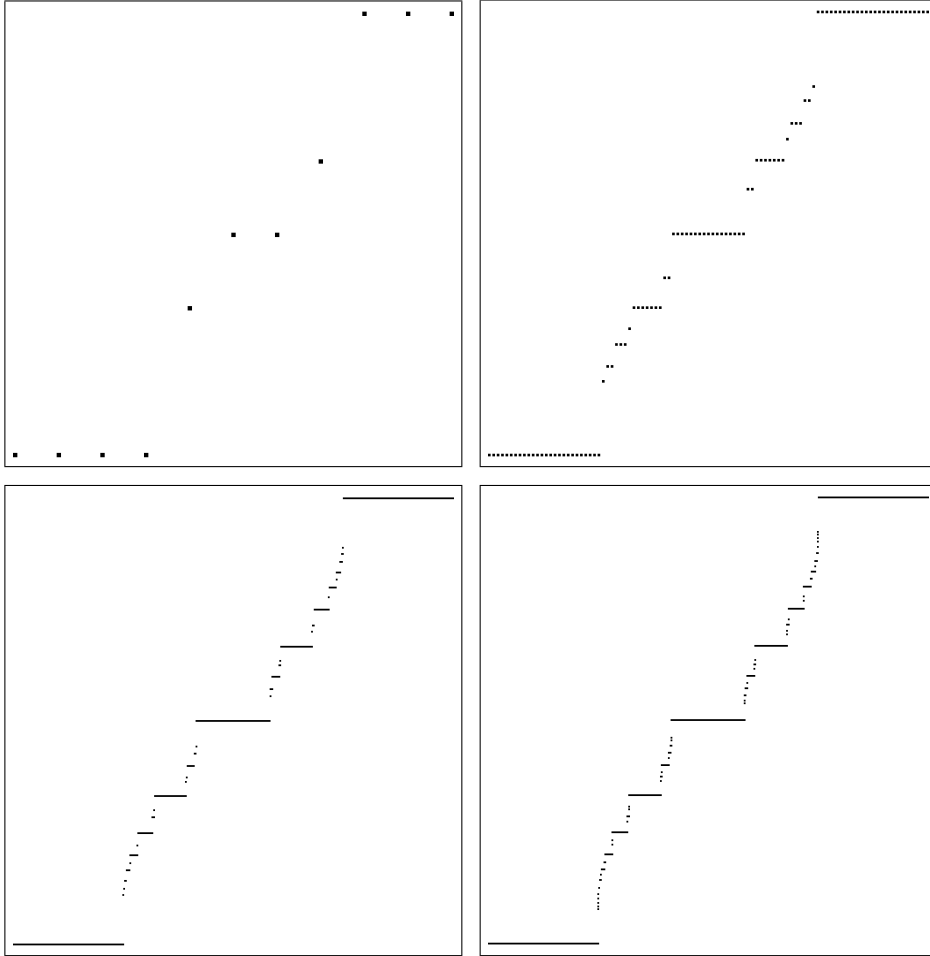


FIGURE 1. The activity phase diagrams $\tilde{s}_n(k) = a(\sigma_n + k)$, for $n = 10$ (top left), 100 (top right), 1000 (bottom left), and 10000, where σ_n is given by (5). On the horizontal axis, k runs from 0 to n . On the vertical axis, \tilde{s}_n runs from 0 to 1.

Related to mode locking, a second feature observed in simulations of parallel chip-firing is non-ergodicity: in trials performed with random initial configurations on the $n \times n$ torus, the activity observed in individual trials differs markedly from the average activity observed over many trials [14]. The experiments of [1] suggested a reason: the chip-firing states in locked modes, corresponding to stairs of the devil's staircase, tend to be periodic with very small period. We study these *short period attractors* in Theorem 17. Under the same hypotheses that yield a devil's staircase in Proposition 10, for each $q \in \mathbb{N}$, a constant fraction $c_q n$ of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q .

staircase of infinitely many preferred modes, there are just three: activity 0, $\frac{1}{2}$ and 1. On the other hand, the numerical experiments of [1] for parallel chip-firing on the $n \times n$ torus suggest a devil's staircase in the large n limit. Our arguments rely quite strongly on the structure of the complete graph, whereas the mode locking phenomenon seems to be widespread. It would be very interesting to relate parallel chip-firing on other graphs to iteration of a circle map (or perhaps a map on a higher-dimensional manifold) in order to explain the ubiquity of mode locking.

2. CONSTRUCTION OF THE CIRCLE MAP

We first identify a certain class of chip configurations on K_n , the *confined states*, with the property that for any configuration σ of activity $a(\sigma) < 1$, we have $U^t\sigma$ confined for all sufficiently large t .

Lemma 1. *For any vertex v , if $\sigma(v) \leq 2n - 1$, then $U\sigma(v) \leq 2n - 1$.*

Proof. If v is unstable in σ , then by (1) we have $U\sigma(v) \leq \sigma(v) \leq 2n - 1$. If v is stable in σ , then $r(\sigma) \leq n - 1$, hence

$$U\sigma(v) \leq \sigma(v) + n - 1 \leq 2n - 2. \quad \square$$

Lemma 2. *If $\sigma(v) \leq 2n - 1$ for all v , then*

$$r \leq U\sigma(v) \leq n - 1 + r$$

for all v , where $r = r(\sigma)$ is the number of unstable vertices in σ .

Proof. The first inequality is immediate from the definition of $U\sigma$. For the second inequality, if v is stable in σ , then $U\sigma(v) \leq n - 1 + r$. If v is unstable in σ , then since $\sigma(v) \leq 2n - 1$, we have

$$U\sigma(v) = \sigma(v) - n + r \leq n - 1 + r. \quad \square$$

Definition. A chip configuration σ on K_n is *preconfined* if it satisfies

- (i) $\sigma(v) \leq 2n - 1$ for all vertices v of K_n .

If, in addition, σ satisfies

- (ii) $\max_v \sigma(v) - \min_v \sigma(v) \leq n - 1$

then σ is *confined*.

By Lemmas 1 and 2, if σ is preconfined, then $U\sigma$ is confined.

Recall (2) that the sequence $(U^t\sigma)_{t \geq 0}$ is eventually periodic. Denote its transient length by t_0 ; that is

$$t_0 = \min\{t \geq 0 \mid U^t\sigma = U^{t'}\sigma \text{ for some } t' > t\}.$$

We say that “ v fires at time t ” if $U^t\sigma(v) \geq n$.

Lemma 3. *If $a(\sigma) < 1$, then $U^t\sigma$ is confined for all $t \geq t_0$.*

Proof. If $a(\sigma) < 1$, then at each time step, some vertex does not fire. If a given vertex v fires at time t , then

$$U^{t+1}\sigma(v) < U^t\sigma(v).$$

Since $U^t\sigma(v) \in \mathbb{Z}_{\geq 0}$ for all t , there is some time T_v at which v does not fire. By Lemma 1, we have $U^t\sigma(v) \leq 2n - 1$ for all $t \geq T_v$. By Lemma 2 we have $U^t\sigma$ confined for all $t > \max_v T_v$. For any $t_1 \geq t_0$ we have $U^t\sigma = U^{t_1}\sigma$ for infinitely many values of t , hence $U^{t_1}\sigma$ is confined. \square

Note that from (1)

$$U\sigma(v) \equiv \sigma(v) + r(\sigma) \pmod{n}.$$

Iterating yields the congruence

$$(7) \quad U^t\sigma(v) \equiv \sigma(v) + \alpha_t \pmod{n}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings before time t .

Our next lemma characterizes the vertices that fire at time $t + 1$.

Lemma 4. *If $U^t\sigma$ is preconfined, then $U^{t+1}\sigma(v) \geq n$ if and only if*

$$\sigma(v) \equiv -j \pmod{n}$$

for some $\alpha_t < j \leq \alpha_{t+1}$.

Proof. By Lemma 2, if $U^t\sigma$ is preconfined, then

$$r_t \leq U^{t+1}\sigma(v) \leq n + r_t - 1$$

for all vertices v , where $r_t = r(U^t\sigma)$. Thus $U^{t+1}\sigma(v) \geq n$ if and only if

$$U^{t+1}\sigma(v) \equiv k \pmod{n}$$

for some $0 \leq k \leq r_t - 1$; by (7), this condition is equivalent to the congruence

$$\sigma(v) + \alpha_{t+1} \equiv k \pmod{n}$$

for some $0 \leq k \leq r_t - 1$. Finally, since $\alpha_{t+1} = \alpha_t + r_t$, we have

$$\{k - \alpha_{t+1} \mid 0 \leq k \leq r_t - 1\} = \{-j \mid \alpha_t < j \leq \alpha_{t+1}\}. \quad \square$$

Let

$$(8) \quad \phi(j) = \#\{v \in [n] \mid \sigma(v) \equiv -j \pmod{n}\}.$$

By Lemma 4, if $U^t\sigma$ is preconfined, then the number of unstable vertices in $U^{t+1}\sigma$ is

$$r_{t+1} = \phi(\alpha_t + 1) + \dots + \phi(\alpha_{t+1}),$$

hence

$$(9) \quad \alpha_{t+2} = \alpha_{t+1} + \sum_{j=\alpha_t+1}^{\alpha_{t+1}} \phi(j).$$

This gives a recurrence for α_t relating three consecutive terms α_t , α_{t+1} and α_{t+2} . Our next lemma simplifies this to a recurrence relating just two consecutive terms.

Lemma 5. *If σ is preconfined, then for all $t \geq 0$*

$$\alpha_{t+1} = g(\alpha_t),$$

where $g : \mathbb{N} \rightarrow \mathbb{N}$ is given by

$$(10) \quad g(k) = \alpha_1 + \sum_{j=1}^k \phi(j)$$

and ϕ is given by (8).

Proof. By Lemma 1, $U^t \sigma$ is preconfined for all $t \geq 0$, so from (9),

$$\begin{aligned} \alpha_{t+1} - \alpha_1 &= \sum_{s=0}^{t-1} (\alpha_{s+2} - \alpha_{s+1}) \\ &= \sum_{s=0}^{t-1} \sum_{j=\alpha_s+1}^{\alpha_{s+1}} \phi(j) \\ &= \sum_{j=1}^{\alpha_t} \phi(j). \quad \square \end{aligned}$$

The function g appearing in Lemma 5 satisfies

$$\begin{aligned} g(k+n) &= g(k) + \sum_{j=k+1}^{k+n} \phi(j) \\ &= g(k) + \sum_{j=k+1}^{k+n} \#\{v \mid \sigma(v) \equiv -j \pmod{n}\} \\ (11) \quad &= g(k) + n. \end{aligned}$$

for all $k \in \mathbb{N}$. This suggests that a more natural domain of definition is the unit circle. First extend g to all of \mathbb{Z} by defining

$$g(-k) = g(nk - k) - nk, \quad k \in \mathbb{N}.$$

This is the unique extension with the property that $g - Id$ is periodic mod n . Now for $x \in \mathbb{R}$, let

$$(12) \quad f(x) = \frac{(1 - \{nx\})g(\lfloor nx \rfloor) + \{nx\}g(\lceil nx \rceil)}{n}$$

where $\lfloor y \rfloor$, $\lceil y \rceil$ and $\{y\}$ denote respectively the greatest integer $\leq y$, the least integer $\geq y$, and the fractional part of y .

By (11) we have for all $x \in \mathbb{R}$

$$\begin{aligned} f(x+1) &= \frac{(1 - \{nx\})g(\lfloor nx \rfloor + n) + \{nx\}g(\lceil nx \rceil + n)}{n} \\ &= f(x) + 1. \end{aligned}$$

Hence the map $f : \mathbb{R} \rightarrow \mathbb{R}$ descends to a circle map $\bar{f} : S^1 \rightarrow S^1$ (viewing S^1 as \mathbb{R}/\mathbb{Z}). Since f is nondecreasing, it has a well-defined *Poincaré rotation number* [8, 12]

$$\rho(f) = \lim_{t \rightarrow \infty} \frac{f^t(x)}{t}$$

which does not depend on x . Here f^t denotes the t -fold iterate $f^t(x) = f(f^{t-1}(x))$, with $f^0 = Id$. The rotation number measures the rate at which the sequence of points $x, \bar{f}(x), \bar{f}(\bar{f}(x)), \dots$ winds around the circle.

Lemma 6. *If σ is preconfined, then $f^t(0) = \alpha_t/n$ for all $t \geq 0$.*

Proof. Induct on t . The base case $t = 0$ is trivial. If $f^t(0) = \alpha_t/n$, then since $\alpha_t \in \mathbb{Z}$, we have by Lemma 5

$$f^{t+1}(0) = f\left(\frac{\alpha_t}{n}\right) = \frac{g(\alpha_t)}{n} = \frac{\alpha_{t+1}}{n}. \quad \square$$

Theorem 7. *If σ is preconfined, then $a(\sigma) = \rho(f)$.*

Proof. By Lemma 6,

$$a(\sigma) = \lim_{t \rightarrow \infty} \frac{\alpha_t}{nt} = \lim_{t \rightarrow \infty} \frac{f^t(0)}{t} = \rho(f). \quad \square$$

Note that the map g is defined in terms of α_1 and ϕ , both of which are easily read off from σ . So given a preconfined configuration σ , equations (10) and (12) give an explicit recipe for writing down a circle map f whose rotation number is the activity of σ .

One naturally wonders how to generalize this construction to chip-firing configurations on graphs other than K_n . A key step may involve identifying invariants of the dynamics. On K_n , these invariants take a very simple form: by (7), for any two vertices $v, w \in [n]$, the difference

$$U^t \sigma(v) - U^t \sigma(w) \pmod n$$

does not depend on t . Analogous invariants for parallel chip-firing on the $n \times n$ torus are classified in [6].

3. DEVIL'S STAIRCASE

Call a map $f : \mathbb{R} \rightarrow \mathbb{R}$ a *monotone degree one lift* if f is continuous, nondecreasing and satisfies

$$(13) \quad f(x+1) = f(x) + 1$$

for all $x \in \mathbb{R}$. Let f, f_n, g be monotone degree one lifts, and denote by $\bar{f}, \bar{f}_n, \bar{g}$ the corresponding circle maps $S^1 \rightarrow S^1$. Write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and $f < g$ if $f(x) < g(x)$ for all $x \in \mathbb{R}$. We will make use of the following well-known properties of the rotation number. For their proofs, see, for example [8, 12].

- **Monotonicity.** If $f \leq g$, then $\rho(f) \leq \rho(g)$.
- **Continuity.** If $\sup |f_n - f| \rightarrow 0$, then $\rho(f_n) \rightarrow \rho(f)$.
- **Conjugation Invariance.** If g is a homeomorphism, then $\rho(g \circ f \circ g^{-1}) = \rho(f)$.
- **Instability of an irrational rotation number.** If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then $\rho(f_1) < \rho(f) < \rho(f_2)$.
- **Stability of a rational rotation number.** If $\rho(f) = p/q \in \mathbb{Q}$, and $\bar{f}^q \neq Id : S^1 \rightarrow S^1$, then for sufficiently small $\epsilon > 0$, either

$$\rho(g) = p/q \text{ whenever } f \leq g \leq f + \epsilon,$$

or

$$\rho(g) = p/q \text{ whenever } f - \epsilon \leq g \leq f.$$

Let $\sigma_2, \sigma_3, \dots$ be a sequence of chip configurations, with σ_n defined on K_n . Suppose σ_n is stable, i.e.,

$$(14) \quad 0 \leq \sigma_n(v) \leq n-1$$

for all $v \in [n]$. Moreover, suppose that there is a continuous function $F : [0, 1] \rightarrow [0, 1]$, such that for all $0 \leq x \leq 1$

$$(15) \quad \frac{1}{n} \#\{v \in [n] \mid \sigma_n(v) < nx\} \rightarrow F(x)$$

as $n \rightarrow \infty$. Since the left side is nondecreasing in x , this convergence is uniform in x . Extend F to all of \mathbb{R} by setting

$$(16) \quad F(x+m) = F(x) + m, \quad m \in \mathbb{Z}.$$

Note that (14) and (15) force $F(0) = 0$ and $F(1) = 1$, so this extension is continuous. By compactness, F is uniformly continuous on $[0, 1]$, so its extension (16) is also uniformly continuous.

The *rescaled activity phase diagram* of σ_n is the function $s_n : [0, 1] \rightarrow [0, 1]$ defined by

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

As $n \rightarrow \infty$, the s_n have a pointwise limit identified in our next result.

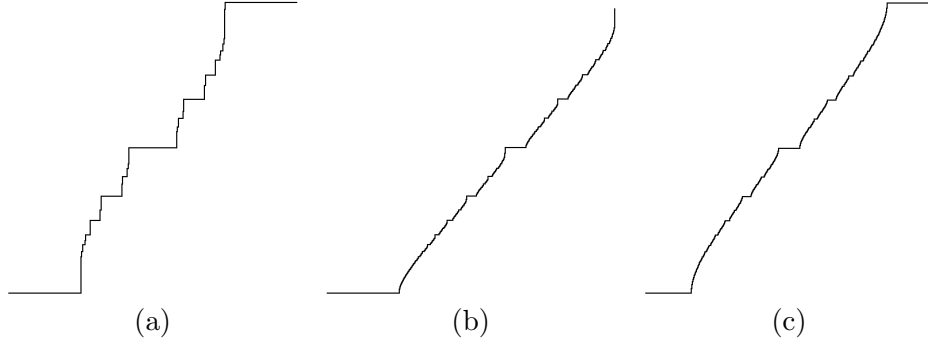


FIGURE 2. The devil's staircase $s(y)$, when (a) $F(x)$ is given by (6); (b) $F(x) = \sqrt{x}$ for $x \in [0, 1]$; and (c) $F(x) = x + \frac{1}{2\pi} \sin 2\pi x$. On the horizontal axis y runs from 0 to 1, and on the vertical axis $s(y)$ runs from 0 to 1.

Theorem 8. *If (14) and (15) hold, then for each $y \in [0, 1]$ we have*

$$s_n(y) \rightarrow s(y) := \rho(R_y \circ \Phi)$$

as $n \rightarrow \infty$, where $\Phi(x) = -F(-x)$, and $R_y(x) = x + y$.

In Proposition 10 we show that under an additional mild hypothesis, the function $s(y)$ is a devil's staircase. Examples of these staircases for different choices of F are shown in Figure 2.

Write

$$\begin{aligned} \phi_{n,y}(j) &= \#\{v \in [n] \mid \sigma_n(v) + \lfloor ny \rfloor \equiv -j \pmod{n}\} \\ (17) \quad &= \phi_{n,0}(j + \lfloor ny \rfloor). \end{aligned}$$

Let $f_{n,y} : \mathbb{R} \rightarrow \mathbb{R}$ be the lift (12) of the circle map associated to the chip configuration $\sigma_n + \lfloor ny \rfloor$. For $x \geq 0$ we have

$$(18) \quad f_{n,y}(x) = \frac{1}{n} \left(r(\sigma_n + \lfloor ny \rfloor) + \sum_{j=1}^{\lfloor nx \rfloor} \phi_{n,y}(j) + \{nx\} \phi_{n,y}(\lceil nx \rceil) \right)$$

where $r(\tau) = \alpha_1(\tau) = \#\{v \in [n] \mid \tau(v) \geq n\}$ is the number of unstable vertices in τ .

Let $\Phi(x) = -F(-x)$.

Lemma 9. *Let $x \in \mathbb{R}$ and $y \in [0, 1]$. If (14) and (15) hold, then as $n \rightarrow \infty$*

$$f_{n,y}(x) \rightarrow \Phi(x + y)$$

uniformly in x and y .

Proof. Since $f_{n,y}(x+1) = f_{n,y}(x) + 1$ for all x , we may assume $x \in [0, 1]$. By (14), a vertex v is unstable in $\sigma_n + \lfloor ny \rfloor$ if and only if $\sigma_n(v) \equiv -j \pmod{n}$

for some $1 \leq j \leq \lfloor ny \rfloor$, so

$$r(\sigma_n + \lfloor ny \rfloor) = \sum_{j=1}^{\lfloor ny \rfloor} \phi_{n,0}(j).$$

Hence from (17) and (18)

$$f_{n,y}(x) = \frac{1}{n} \left(\sum_{j=1}^{\lfloor nx \rfloor + \lfloor ny \rfloor} \phi_{n,0}(j) + \{nx\} \phi_{n,0}(\lfloor nx \rfloor + \lfloor ny \rfloor) \right).$$

As $\lfloor nx \rfloor + \lfloor ny \rfloor$ is either equal to or one less than $\lfloor n(x+y) \rfloor$, we obtain

$$(19) \quad \frac{1}{n} \sum_{j=1}^{\lfloor n(x+y) \rfloor - 1} \phi_{n,0}(j) \leq f_{n,y}(x) \leq \frac{1}{n} \sum_{j=1}^{\lfloor n(x+y) \rfloor} \phi_{n,0}(j).$$

Since $0 \leq \sigma_n(v) \leq n-1$, we have

$$\phi_{n,0}(j) = \#\{v \in [n] \mid \sigma_n(v) = n-j\}.$$

If $x+y \leq 1$, then

$$\begin{aligned} \sum_{j=1}^{\lfloor n(x+y) \rfloor} \phi_{n,0}(j) &= \#\{v \in [n] \mid \sigma_n(v) \geq n(1-x-y)\} \\ &= n - \#\{v \in [n] \mid \sigma_n(v) < n(1-x-y)\}. \end{aligned}$$

From (15) we obtain

$$(20) \quad \frac{1}{n} \sum_{j=1}^{\lfloor n(x+y) \rfloor} \phi_{n,0}(j) \rightarrow 1 - F(1-x-y) = \Phi(x+y)$$

uniformly in x and y , as $n \rightarrow \infty$. Likewise, if $x+y > 1$, then since $\sum_{j=1}^n \phi_{n,0}(j) = n$, we have from (20)

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^{\lfloor n(x+y) \rfloor} \phi_{n,0}(j) &= 1 + \frac{1}{n} \sum_{j=1}^{\lfloor n(x+y-1) \rfloor} \phi_{n,0}(j) \\ &\rightarrow 1 + \Phi(x+y-1) = \Phi(x+y) \end{aligned}$$

uniformly in x and y , as $n \rightarrow \infty$.

Fix $\epsilon > 0$. From (19), we have

$$\Phi(x+y-\epsilon) - \epsilon \leq f_{n,y}(x) \leq \Phi(x+y+\epsilon) + \epsilon$$

for all sufficiently large n . Since Φ is uniformly continuous, we conclude that

$$f_{n,y}(x) \rightarrow \Phi(x+y)$$

uniformly in x and y . □

Proof of Theorem 8. By (14), the configuration $\sigma_n + \lfloor ny \rfloor$ is confined, so by Theorem 7 we have

$$s_n(y) = \rho(f_{n,y})$$

where $f_{n,y}$ is given by (18). By Lemma 9 we have $\sup |f_{n,y} - \Phi \circ R_y| \rightarrow 0$. By the continuity of the rotation number, we conclude that

$$s_n(y) = \rho(f_{n,y}) \rightarrow \rho(\Phi \circ R_y) = \rho(R_y \circ \Phi)$$

as $n \rightarrow \infty$, where in the last equality we have used the conjugation invariance of the rotation number. \square

Write $\Phi_y = R_y \circ \Phi$, and let $\bar{\Phi}_y : S^1 \rightarrow S^1$ be the corresponding circle map. We will call a function $s : [0, 1] \rightarrow [0, 1]$ a *devil's staircase* if it is continuous, nondecreasing, nonconstant, and locally constant on an open dense set. Next we show that if

$$(21) \quad (\bar{\Phi}_y)^q \neq Id \quad \text{for all } y \in S^1 \text{ and all } q \in \mathbb{N},$$

then the limiting function $s(y)$ in Theorem 8 is a devil's staircase.

Proposition 10. *The function $s(y) = \rho(\Phi_y)$ continuous and nondecreasing in y . If $z \in [0, 1]$ is irrational, then $s^{-1}(z)$ is a point. Moreover, if (21) holds, then for every rational number $p/q \in [0, 1]$ the fiber $s^{-1}(p/q)$ is an interval of positive length.*

Substantially similar results appear in [8, 12]; we include a proof here for the sake of completeness.

Proof. The monotonicity of the rotation number implies that s is nondecreasing. Since $\sup |\Phi_y - \Phi_{y'}| = |y - y'|$, the continuity of the rotation number implies s is continuous.

Since $\Phi(0) = 0$, we have $s(0) = 0$ and $s(1) = 1$. By the intermediate value theorem, $s : [0, 1] \rightarrow [0, 1]$ is onto. If $z = s(y)$ is irrational, and $y_1 < y < y_2$, then $\Phi_{y_1} < \Phi_y < \Phi_{y_2}$, hence

$$s(y_1) < s(y) < s(y_2)$$

by the instability of an irrational rotation number. It follows that $s^{-1}(z) = \{y\}$.

If $s(y) = p/q \in \mathbb{Q}$, then by the stability of a rational rotation number, since $(\bar{\Phi}_y)^q \neq Id$, there exists an interval I of positive length (take either $I = [y - \epsilon, y]$ or $I = [y, y + \epsilon]$ for small enough ϵ) such that $\rho(\Phi_{y'}) = p/q$ for all $y' \in I$. Hence $I \subset s^{-1}(p/q)$. \square

Note that if $y \leq y'$, then $f_{n,y} \leq f_{n,y'}$, so $s_n(y) = \rho(f_{n,y})$ is nondecreasing in y . By the continuity of s , it follows that the convergence in Theorem 8 is uniform in y .

Our next result shows that in the interiors of the stairs, we in fact have $s_n(y) = s(y)$ for sufficiently large n .

Proposition 11. *Suppose that (14), (15) and (21) hold. If $s^{-1}(p/q) = [a, b]$, then for any $\epsilon > 0$*

$$[a + \epsilon, b - \epsilon] \subset s_n^{-1}(p/q)$$

for all sufficiently large n .

Proof. Fix $\epsilon > 0$. By Lemma 9, for sufficiently large n we have

$$\sup |f_{n,y} - \Phi \circ R_y| < \epsilon$$

for all $y \in [0, 1]$. Then $f_{n,y} \geq R_{-\epsilon} \circ \Phi \circ R_y$, so for any $y \geq a + \epsilon$ we have by the monotonicity and conjugation invariance of the rotation number

$$\begin{aligned} s_n(y) &= \rho(f_{n,y}) \geq \rho(R_{-\epsilon} \circ \Phi \circ R_y) \\ &= \rho(R_{y-\epsilon} \circ \Phi) = s(y - \epsilon) \geq s(a) = p/q. \end{aligned}$$

Likewise $f_{n,y} \leq R_\epsilon \circ \Phi \circ R_y$, hence for any $y \leq b - \epsilon$

$$\begin{aligned} s_n(y) &= \rho(f_{n,y}) \leq \rho(R_\epsilon \circ \Phi \circ R_y) \\ &= \rho(R_{y+\epsilon} \circ \Phi) = s(y + \epsilon) \leq s(b) = p/q. \quad \square \end{aligned}$$

4. SHORT PERIOD ATTRACTORS

For a chip configuration σ on K_n and a vertex $v \in [n]$, let

$$u_t(\sigma, v) = \#\{0 \leq s < t \mid U^s \sigma(v) \geq n\}$$

be the number of times v fires during the first t updates. During these updates, the vertex v emits a total of $n u_t(\sigma, v)$ chips and receives a total of $\alpha_t = \sum_w u_t(\sigma, w)$ chips, so that

$$(22) \quad U^t \sigma(v) - \sigma(v) = \alpha_t - n u_t(\sigma, v).$$

An easy consequence that we will use repeatedly is the following.

Lemma 12. *A chip configuration σ on K_n satisfies $U^t \sigma = \sigma$ if and only if*

$$(23) \quad u_t(\sigma, v) = u_t(\sigma, w)$$

for all vertices v and w .

Proof. If (23) holds, then $u_t(\sigma, v) = \alpha_t/n$ for all v , so $U^t \sigma(v) = \sigma(v)$ by (22). Conversely, if $U^t \sigma = \sigma$, then the left side of (22) vanishes, so $u_t(\sigma, v) = \alpha_t/n$ for all vertices v . \square

According to our next lemma, if σ is confined, then $u_t(\sigma, v)$ and $u_t(\sigma, w)$ differ by at most one.

Lemma 13. *If σ is confined, and $\sigma(v) \leq \sigma(w)$, then for all $t \geq 0$*

$$u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1.$$

Proof. Induct on t . If $u_t(\sigma, w) = u_t(\sigma, v)$, then by (22)

$$U^t\sigma(v) - U^t\sigma(w) = \sigma(v) - \sigma(w) \leq 0.$$

Thus if v fires at time t , then w must also fire, hence

$$(24) \quad u_{t+1}(\sigma, v) \leq u_{t+1}(\sigma, w) \leq u_{t+1}(\sigma, v) + 1.$$

On the other hand, if $u_t(\sigma, w) = u_t(\sigma, v) + 1$, then since σ is confined, we have by (22)

$$U^t\sigma(v) - U^t\sigma(w) = n + \sigma(v) - \sigma(w) \geq 0.$$

Thus if w fires at time t , then v must also fire, so once again (24) holds. \square

Lemma 14. *If σ is confined, then $U^t\sigma = \sigma$ if and only if $n|\alpha_t$.*

Proof. If $U^t\sigma = \sigma$, then $n|\alpha_t$ by Lemma 12. For the converse, write $\ell = \min_v u_t(\sigma, v)$. By Lemma 13 we have

$$u_t(\sigma, v) \in \{\ell, \ell + 1\}$$

for all vertices v . If $n|\alpha_t$, then since

$$\alpha_t = \sum_{v=1}^n u_t(\sigma, v),$$

we must have $u_t(\sigma, v) = \ell$ for all v , so $U^t\sigma = \sigma$ Lemma 12. \square

Let σ be a confined state on K_n . By the pigeonhole principle, there exist times $0 \leq s < t \leq n$ with

$$\alpha_s \equiv \alpha_t \pmod{n}.$$

By Lemma 14 it follows that $U^s\sigma = U^t\sigma$, so σ has eventual period at most n .

Our next result improves this bound a bit. Write $m(\sigma)$ for the eventual period of σ , and

$$\nu(\sigma) = \#\{\sigma(v) | v \in [n]\}$$

for the number of distinct heights in σ .

Proposition 15. *For any chip configuration σ on K_n ,*

$$m(\sigma) \leq \nu(\sigma).$$

Proof. If $a(\sigma) = 1$, then $m(\sigma) = 1$. Since $\nu(U\sigma) \leq \nu(\sigma)$ and $m(U\sigma) = m(\sigma)$, by Lemma 3 it suffices to prove the proposition for confined states σ . Moreover, by relabeling the vertices we may assume that $\sigma(1) \geq \sigma(2) \geq \dots \geq \sigma(n)$. Write the total activity α_t as

$$\alpha_t = Q_t n + R_t$$

for nonnegative integers Q_t and R_t with $R_t \leq n - 1$. By Lemma 13, since $\sum_v u_t(\sigma, v) = \alpha_t$, we must have

$$u_t(\sigma, v) = \begin{cases} Q_t + 1, & v \leq R_t \\ Q_t, & v > R_t. \end{cases}$$

Moreover if $R_t > 0$, then $\sigma(R_t) > \sigma(R_t + 1)$. Since the set

$$V = \{0\} \cup \{v \in [n-1] \mid \sigma(v) > \sigma(v+1)\}$$

has cardinality $\nu(\sigma)$, by the pigeonhole principle there exist times $0 \leq s < t \leq \nu(\sigma)$ with $R_s = R_t$. Then

$$u_t(\sigma, v) - u_s(\sigma, v) = Q_t - Q_s$$

independent of v , hence $U^s \sigma = U^t \sigma$ by Lemma 12. Thus $m(\sigma) \leq t - s \leq \nu(\sigma)$. \square

Bitar [3] conjectured that any parallel chip-firing configuration on a connected graph of n vertices has eventual period at most n . A counterexample was found by Kiwi et al. [10]. It would be interesting to investigate for what classes of graphs Bitar's conjecture does hold; for example, no counterexample seems to be known on a regular graph.

Next we relate the period to the activity. We will use the fact that the rotation number of a circle map determines the periods of its periodic points: if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a monotone degree one lift (13) with $\rho(f) = p/q$ in lowest terms, then all periodic points of $\bar{f} : S^1 \rightarrow S^1$ have period q ; see Proposition 4.3.8 and Exercise 4.3.5 of [8].

Note that if a chip configuration σ on K_n is periodic of period m , and $a(\sigma) = p/q$, then

$$(25) \quad \frac{\alpha_m}{mn} = \frac{p}{q}$$

and $\alpha_{km} = k\alpha_m$ for all $k \in \mathbb{N}$.

Lemma 16. *If $a(\sigma) = p/q$ and $(p, q) = 1$, then $m(\sigma) = q$.*

Proof. Since $a(U\sigma) = a(\sigma)$ and $m(U\sigma) = m(\sigma)$, we may assume that σ is periodic. Now if $a(\sigma) = 1$, then $m(\sigma) = 1$, so we may assume $a(\sigma) < 1$. In particular, σ is confined by Lemma 3. Write $m = m(\sigma)$. By Lemma 12 and (25)

$$u_m(\sigma, v) = \frac{\alpha_m}{n} = \frac{pm}{q}$$

for all vertices v . Hence $q \mid pm$ and hence $q \mid m$.

Let f be the lift (12) of the circle map \bar{f} constructed in section 2. By Theorem 7 we have $\rho(f) = a(\sigma)$, so all periodic points of \bar{f} have period q . By Lemma 6 we have

$$f^{qm}(0) = \frac{\alpha_{qm}}{n} = \frac{q\alpha_m}{n} = pm \in \mathbb{Z},$$

so 0 is a periodic point of \bar{f} . Thus $f^q(0) \in \mathbb{Z}$. Now by Lemma 6

$$\alpha_q = n f^q(0)$$

hence $U^q \sigma = \sigma$ by Lemma 14. Thus $m \mid q$. \square

Given $1 \leq p < q \leq n$ with $(p, q) = 1$ and $p/q \leq 1/2$, one can check that the chip configuration σ on K_n given by

$$\sigma(v) = \begin{cases} v + p - 1, & v \leq q - 1 - p \\ v + n + p - q - 1, & q - p \leq v \leq q - 1 \\ n + p - 1, & v \geq q. \end{cases}$$

has activity $a(\sigma) = p/q$. For a similar construction on more general graphs in the case $p = 1$, see [13, Prop. 6.8]. By Lemma 16, $m(\sigma) = q$. So for every integer $q = 1, \dots, n$ there exists a chip configuration on K_n of period q .

Despite the existence of states of period as large as n , states of smaller period are in some sense more prevalent. One way to capture this is the following.

Theorem 17. *If $\sigma_2, \sigma_3, \dots$ is a sequence of chip configurations satisfying (14), (15) and (21), then for each $q \in \mathbb{N}$ there is a constant $c = c_q > 0$ such that for all sufficiently large n , at least cn of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q .*

Proof. Fix $\epsilon > 0$, and choose $p < q$ with $(p, q) = 1$. By Proposition 10 we have $s^{-1}(p/q) = [a_p, b_p]$ for some $a_p < b_p$. By Proposition 11, for sufficiently large n we have

$$a(\sigma_n + \lfloor ny \rfloor) = s_n(y) = p/q$$

for all $y \in [a_p + \epsilon/2q, b_p - \epsilon/2q]$. Thus all states $\sigma_n + k$ with

$$\left(a_p + \frac{\epsilon}{2q}\right)n < k < \left(b_p - \frac{\epsilon}{2q}\right)n$$

have activity p/q and hence, by Lemma 16, eventual period q . We can therefore take

$$c_q = \sum_{p:(p,q)=1} (b_p - a_p) - \epsilon$$

for any $\epsilon > 0$. □

The rest of this section is devoted to proving the existence of a *period 2 window*: any chip configuration on K_n with total number of chips strictly between $n^2 - n$ and n^2 has eventual period 2.

The following lemma is a special case of [13, Prop. 6.2]; we include a proof for completeness. Write $r(\sigma)$ for the number of unstable vertices in σ .

Lemma 18. *If σ and τ are chip configurations on K_n with $\sigma(v) + \tau(v) = 2n - 1$ for all v , then $a(\sigma) + a(\tau) = 1$.*

Proof. If $\sigma(v) + \tau(v) = 2n - 1$ for all v , then for each vertex v , exactly one of $\sigma(v), \tau(v)$ is $\geq n$. Hence $r(\sigma) + r(\tau) = n$, and

$$\begin{aligned} U\sigma(v) + U\tau(v) &= \sigma(v) + r(\sigma) + \tau(v) + r(\tau) - n \\ &= 2n - 1 \end{aligned}$$

for all vertices v . Inducting on t , we obtain

$$r(U^t\sigma) + r(U^t\tau) = n$$

for all $t \geq 0$, and hence $a(\sigma) + a(\tau) = 1$ by (3). \square

Given a chip configuration σ on K_n , for $j = 1, \dots, n$ we define *conjugate* configurations

$$c^j\sigma(v) = \begin{cases} \sigma(v) + j - n, & v \leq j \\ \sigma(v) + j, & v > j. \end{cases}$$

Note that $c^n\sigma = \sigma$.

Lemma 19. *Let σ be a chip configuration on K_n , and fix $j \in [n]$. For all $t \geq 0$, we have for $v \leq j$*

$$u_t(\sigma, v) - 1 \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v),$$

while for $v > j$

$$u_t(\sigma, v) \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v) + 1.$$

Proof. Induct on t . By (22)

$$(26) \quad U^t c^j\sigma(v) = c^j\sigma(v) + \alpha_{t,j} - n u_t(c^j\sigma, v)$$

where

$$\alpha_{t,j} = \sum_{w=1}^n u_t(c^j\sigma, w).$$

By the inductive hypothesis,

$$(27) \quad -j \leq \alpha_{t,j} - \alpha_{t,0} \leq n - j.$$

Fix a vertex v , and write

$$b = u_t(c^j\sigma, v) - u_t(\sigma, v).$$

By the inductive hypothesis we have $b \in \{-1, 0, 1\}$. From (26) we have

$$(28) \quad U^t c^j\sigma(v) - U^t\sigma(v) = \alpha_{t,j} - \alpha_{t,0} + c^j\sigma(v) - \sigma(v) - nb$$

Case 1: $v \leq j$ and $b = -1$. By (27), the right side of (28) is ≥ 0 , so if v is unstable in $U^t\sigma$, then it is also unstable in $U^t c^j\sigma$. Thus

$$(29) \quad u_{t+1}(\sigma, v) - 1 \leq u_{t+1}(c^j\sigma, v) \leq u_{t+1}(\sigma, v),$$

completing the inductive step.

Case 2: $v \leq j$ and $b = 0$. By (27), the right side of (28) is ≤ 0 , so if v is unstable in $U^t c^j\sigma$, then it is also unstable in $U^t\sigma$. Thus once again (29) holds.

Case 3: $v > j$ and $b = 0$. By (27), the right side of (28) is ≥ 0 , so

$$(30) \quad u_{t+1}(\sigma, v) \leq u_{t+1}(c^j\sigma, v) \leq u_{t+1}(\sigma, v) + 1,$$

completing the inductive step.

Case 4: $v > j$ and $b = 1$. By (27), the right side of (28) is ≤ 0 , so once again (30) holds. \square

Corollary 20. *For any chip configuration σ on K_n and any $j \in [n]$,*

$$a(c^j \sigma) = a(\sigma).$$

It turns out that the circle maps corresponding to σ and $c^j \sigma$ are conjugate to one another by a rotation. This gives an alternative proof of the corollary, in the case when both σ and $c^j \sigma$ are preconfined.

Lemma 21. *Let σ be a chip configuration on K_n . If $u_2(\sigma, v) \geq 1$ for all v , then $u_{2t}(\sigma, v) \geq t$ for all v and all $t \geq 1$.*

Proof. Induct on t . By the inductive hypothesis, $\alpha_{2t} \geq nt$. Fix a vertex v , and suppose that $u_{2t}(\sigma, v) = t$. Since $u_2(\sigma, v) \geq 1$, either $\sigma(v) \geq n$ or $U\sigma(v) \geq n$. In the former case, by (22)

$$U^{2t}\sigma(v) - \sigma(v) = \alpha_{2t} - nt \geq 0,$$

so v fires at time $2t$. Hence $u_{2t+1}(\sigma, v) \geq t + 1$. Summing over v yields

$$\alpha_{2t+1} \geq \alpha_1(t+1) + (n - \alpha_1)t = nt + \alpha_1.$$

In the latter case, if v does not fire at time $2t$, we have by (22)

$$U^{2t+1}\sigma(v) - U\sigma(v) = \alpha_{2t+1} - \alpha_1 - nt \geq 0,$$

so v fires at time $2t + 1$. Hence $u_{2t+2}(\sigma, v) \geq t + 1$ for all v . \square

Given a chip configuration σ on K_n , write

$$\ell(\sigma) = \min\{\sigma(1), \dots, \sigma(n)\}$$

and

$$r(\sigma) = \#\{v \in [n] : \sigma(v) \geq n\}.$$

Write

$$|\sigma| = \sum_{v=1}^n \sigma(v)$$

for the total number of chips in the system.

Theorem 22. *Every chip configuration σ on K_n with $n^2 - n < |\sigma| < n^2$ has eventual period 2.*

Proof. If $|\sigma| < n^2$, then $a(\sigma) < 1$, so by Lemma 3 we may assume σ is confined. Moreover, by relabeling the vertices we may assume that $\sigma(1) \geq \sigma(2) \geq \dots \geq \sigma(n)$. In particular, for any vertex v , since $\sigma(w) \geq \sigma(v) - n + 1$ for $w = v + 1, \dots, n$, we have

$$\begin{aligned} |\sigma| &\geq v\sigma(v) + (n-v)(\sigma(v) - n + 1) \\ &= n\sigma(v) - (n-v)(n-1). \end{aligned}$$

Since $|\sigma| < n^2$, we obtain

$$(31) \quad \sigma(v) < \frac{n^2 + (n-v)(n-1)}{n} \leq 2n - v.$$

For a fixed vertex v , as j ranges over $[n]$, the quantity $c^j\sigma(v)$ takes on each of the values

$$\sigma(v) + v - n, \dots, \sigma(v) + v - 1$$

exactly once. By (31), at least $\sigma(v) + v - n$ of these values are $\geq n$, hence

$$(32) \quad \sum_{j=1}^n r(c^j\sigma) \geq \sum_{v=1}^n (\sigma(v) + v - n) = |\sigma| - \frac{n(n-1)}{2}.$$

Since $\ell(c^j\sigma) = \sigma(j) + j - n$, we have

$$(33) \quad \sum_{j=1}^n \ell(c^j\sigma) = |\sigma| - \frac{n(n-1)}{2}.$$

Summing equations (32) and (33), and using $|\sigma| > n^2 - n$, we obtain

$$\sum_{j=1}^n (\ell(c^j\sigma) + r(c^j\sigma)) > n^2 - n.$$

Since each term in the sum on the left is a nonnegative integer, we must have

$$\ell(c^j\sigma) + r(c^j\sigma) \geq n.$$

for some $j \in [n]$. Thus every vertex v fires at least once during the first two updates of $c^j\sigma$. From Corollary 20 and Lemma 21 we obtain

$$a(\sigma) = a(c^j\sigma) \geq \frac{1}{2}.$$

Finally, the chip configuration $\tau(v) := 2n - 1 - \sigma(v)$ also satisfies $n^2 - n < |\tau| < n^2$, so $a(\tau) \geq \frac{1}{2}$. By Lemma 18 we have $a(\sigma) + a(\tau) = 1$, so $a(\sigma) = a(\tau) = \frac{1}{2}$. From Lemma 16 we conclude that $m(\sigma) = 2$. \square

ACKNOWLEDGEMENT

The author thanks Anne Fey for many helpful conversations.

REFERENCES

- [1] F. Bagnoli, F. Cecconi, A. Flammini, and A. Vespignani, Short-period attractors and non-ergodic behavior in the deterministic fixed-energy sandpile model, *Europhys. Lett.* **63** (2003), 512–518.
- [2] P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality: an explanation of the $1/f$ noise, *Phys. Rev. Lett.* **59**, no. 4 (1987), 381–384.
- [3] J. Bitar, Juegos Combinatoriales en Redes de Autómatas, Tesis Ingeniero Matemático, Dept. Ingeniería Matemática, U. de Chile, 1989.
- [4] J. Bitar and E. Goles, Parallel chip firing games on graphs, *Theoretical Comp. Sci.* **92**, no. 2 (1992), 291–300.
- [5] A. Björner, L. Lovász and P. Shor, Chip-firing games on graphs, *European J. Combin.* **12** (1991), no. 4, 283–291.
- [6] M. Casartelli, L. Dall’Asta, A. Vezzani and P. Vivo, Dynamical invariants in the deterministic fixed-energy sandpile, *European Phys. J. B* **52** (2006), no. 1, 91–105.
- [7] L. Dall’Asta, Exact solution of the one-dimensional deterministic fixed-energy sandpile, *Phys. Rev. Lett.* **96** (2006).

- [8] B. Hasselblatt and A. Katok, *A First Course in Dynamics, With a Panorama of Recent Developments*, Cambridge University Press, 2003.
- [9] S. A. Janowsky and C. A. Laberge, Exact solutions for a mean-field abelian sandpile, *J. Phys. A* **26** (1993) L973–L980.
- [10] M. A. Kiwi, R. Ndoundam, M. Tchuente and E. Goles, No polynomial bound for the period of the parallel chip firing game on graphs, *Theoretical Comp. Sci.* **136**, no. 2 (1994), 527–532.
- [11] J. C. Lagarias, Number theory and dynamical systems, *Proc. Symp. Applied Math.* **46** (1992), 35–72.
- [12] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1991.
- [13] E. Prisner, Parallel chip-firing on digraphs, *Complex Systems* **8** (1994), 367–383.
- [14] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Absorbing-state phase transitions in fixed-energy sandpiles, *Phys. Rev. E* **62** (2000), 4564–4582.

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