EXAMPLES OF PLANAR TIGHT CONTACT STRUCTURES WITH SUPPORT NORM ONE

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ABSTRACT. We exhibit an infinite family of tight contact structures none of which is supported by an open book whose pages have minimal genus and maximum Euler characteristic among those of all open books supporting the same contact structure, answering a question of Baldwin and Etnyre in [2].

Let Y be a closed oriented 3–manifold and ξ be a contact structure on Y. It is well known that ξ is supported by an open book on Y and all open book decompositions of Y supporting ξ are equivalent up to positive stabilizations and destabilizations [5]. In [4], Etnyre and Ozbagci define three numerical invariants of ξ, called the *support norm, support genus* and *binding number*, respectively, in terms of its supporting open books:

$$\mathrm{sn}(\xi) = \min\{-\chi(\pi^{-1}(\theta)|\pi : Y - B \to S^1 \text{ supports } \xi\}$$
$$\mathrm{sg}(\xi) = \min\{g(\pi^{-1}(\theta))|\pi : Y - B \to S^1 \text{ supports } \xi\}$$
$$\mathrm{bn}(\xi) = \min\{|B||\pi : Y - B \to S^1 \text{ supports } \xi \text{ and } \ g(\pi^{-1}(\theta)) = sg(\xi)\}\ ,$$

where θ is any point in S^1 , $g(.)$ is the genus, and $|.|$ is the number of components. In general, these invariants are hard to compute. It is known that $\mathrm{sg}(\xi) = 0$ if ξ is overtwisted [3], and in general there are obstructions for a contact structure to have support genus zero ([3, 9]). However, there is no known example of a contact structure with support genus greater than one. Even if ξ is overtwisted, it is not easy to determine $\mathrm{bn}(\xi)$. Furthermore, it is known that no two of these invariants determine the third [2].

It is obvious from the above definitions that

$$\mathrm{sn}(\xi) \leq 2\mathrm{sg}(\xi) + \mathrm{bn}(\xi) - 2\ ,$$

and that equality holds when $\mathrm{bn}(\xi) \leq 3$.

In [2], Baldwin and Etnyre exhibit examples of *overtwisted* contact structures which make the above inequality strict and ask whether the inequality can be strict for tight contact structures. Here we give an infinite family of tight contact structures (exactly one of which is Stein fillable) for which this inequality is strict.

Let T_0 be genus one surface with one boundary component and consider the family of diffeomorphisms $\phi_m = (\tau_a\tau_b)^3\tau_a^{-m-4}$, for $m \geq 0$, where a and b are simple closed curves given in Figure 1 and τ denotes the right-handed Dehn twist along the corresponding curve.

We would like to thank Selman Akbulut and Burak Ozbagci for helpful conversations. TE was partially supported by the Scientific and Technological Research Council of Turkey.

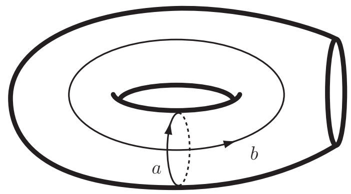

FIGURE 1.

For later use, we orient a and b so that $a \cdot b = -1$. Let (Y_m, ξ_m) denote the contact manifold supported by the open book decomposition (T_0, ϕ_m).

Theorem. *The contact structure ξ_0 is Stein fillable and ξ_m is tight but not Stein fillable for $m > 0$. Furthermore,*

$$sn(\xi_m) = 1\ ,\ sg(\xi_m) = 0\ ,\ 3 < bn(\xi_m) \leq m + 5$$

In particular, $sn(\xi_m) < 2sg(\xi_m) + bn(\xi_m) - 2$.

Proof. The fact that every ξ_m is tight with nontrivial Heegaard Floer invariant $c(Y_m, \xi_m) \in \widehat{HF}(-Y_m, \mathfrak{s}_{\xi_m})$ follows from Theorem 4.3 in [6]. Now, observe that $\phi_0 = (\tau_a \tau_b)^3 \tau_a^{-4} = \tau_{a+b}\tau_{a-b}$. Since it is supported by an open book whose monodromy is a product of right-handed Dehn twists, ξ_0 is Stein fillable. In general, we have $\phi_m = \tau_{a+b}\tau_{a-b}\tau_a^{-m}$. Using this factorization, we draw a handlebody diagram of a 4–manifold X_m with boundary Y_m in Figure 2.

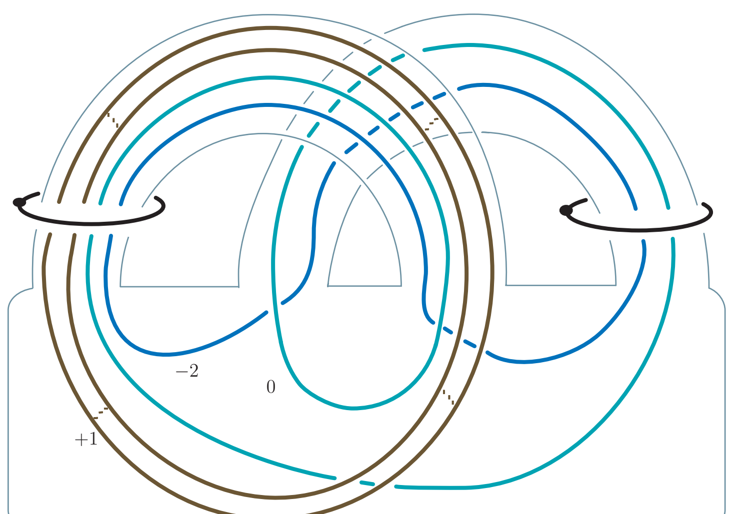

FIGURE 2. Handlebody diagram of X_m with two 1-handles, m +1-framed 2-handles, a -2-framed 2-handle, and a 0-framed 2-handle

Figure 3 describes a way to see that Y_m is diffeomorphic to the Seifert fibered 3–manifold $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{m+2})$. A complete classification of tight contact structures on $M(-1; \frac{1}{2}, \frac{1}{2}, \frac{1}{m+2})$

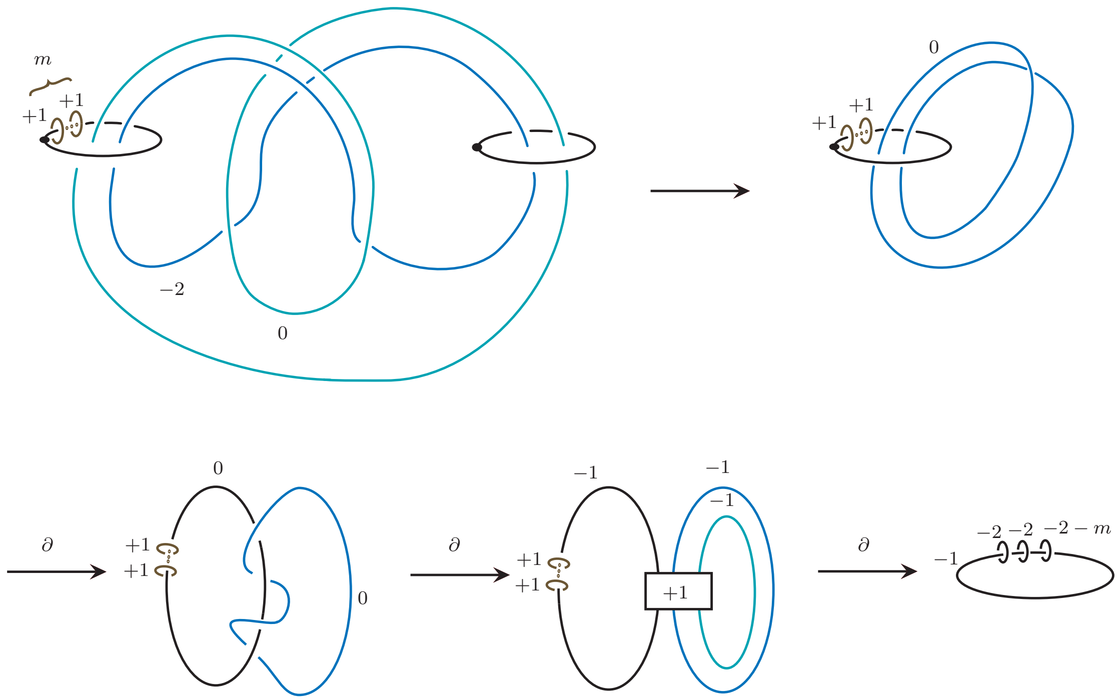

FIGURE 3. Seifert fibered 3–manifold description of Y_m

is given in Section 4 of [8]. It follows, in particular, that all the tight contact structures on these manifolds are supported by planar open books, i.e. $\mathrm{sg}(\xi_m) = 0$. In fact, we can pinpoint precisely the contact isotopy class of ξ_m from this classification by calculating a Hopf invariant, $d_3(\xi_m)$. Indeed, in Theorem 4.1 and Proposition 5.1 of [1] Baldwin shows that Y_m is an L-space and calculates the correction term $d(Y_m, \mathfrak{s}_{\xi_m}) = -\frac{m}{4}$. Since we also know that $c(Y_m, \xi_m)$ is non-zero, and it has grading equal to $-d_3(\xi_m)$ [1] by Proposition 4.6 of [10], we conclude that $d_3(\xi_m) = d(Y_m, \mathfrak{s}_{\xi_m}) = -\frac{m}{4}$. (Note that this calculation can also be done by drawing a contact surgery diagram associated with ϕ_m). For each $m > 0$ there are three tight contact structures on Y_m [8] exactly one of which has d_3 invariant equal to $-\frac{m}{4}$, and it is given by the contact surgery diagram in Figure 4. Note that, the fact that ξ_m is Stein fillable if and only if $m = 0$ follows from Theorem 4.13 in [8].

So far, we have shown that $\mathrm{sg}(\xi_m) = 0$, and $\mathrm{sn}(\xi_m) \leq 1$, where the latter follows because we started with an open book supporting ξ_m with pages a genus one surface with one boundary component. Furthermore, Figure 4 gives a planar open book supporting ξ_m with $m+5$ boundary components, hence $\mathrm{bn}(\xi_m) \leq m+5$. Next, observe that $\mathrm{sn}(\xi) < 1$ implies that ξ is a contact structure on a lens space. Hence $\mathrm{sn}(\xi_m) = 1$. To finish, we need to show that $\mathrm{bn}(\xi_m) \neq 3$. Any 3–manifold with a planar open book with three binding components

[1] Here we follow the convention in [8] where the Hopf invariant is shifted by $1/2$ so that it is 0 for the standard contact structure on S^3.

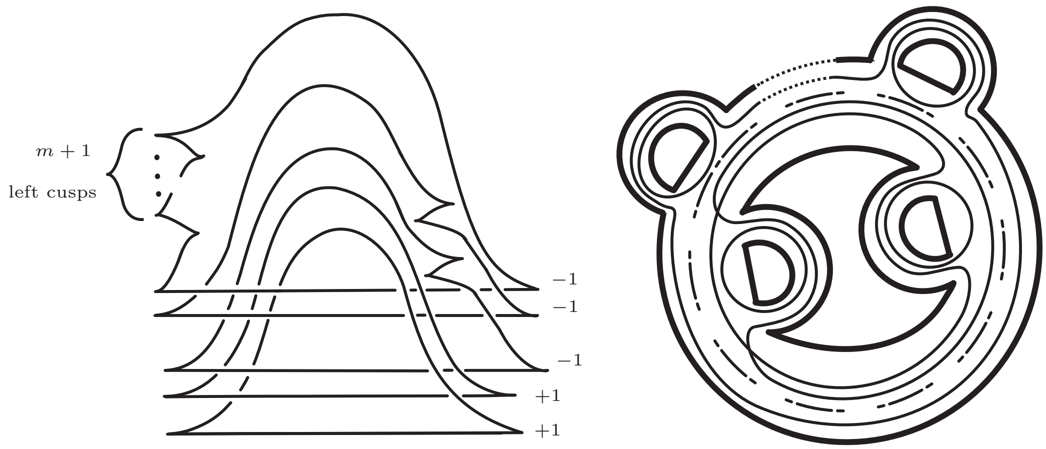

FIGURE 4. On the left: Contact surgery diagram of ξ_m as given by Figure 7 in [8]. On the right: A planar open book supporting ξ_m with $m+5$ boundary components, the monodromy is negative Dehn twist around the middle dashed curve and positive Dehn twist around all the other curves.

is given by a surgery diagram as in Figure 5. These are connected sums of lens spaces if $\{0, \pm 1\} \cap \{p, q, r\} \neq \emptyset$, and small Seifert fibered spaces with $e_0 = \lfloor -\frac{1}{p} \rfloor + \lfloor -\frac{1}{q} \rfloor + \lfloor -\frac{1}{r} \rfloor$ otherwise. Since $e_0(Y_m) = -1$ any open book decomposition of Y_m with planar pages and three binding components must have exactly two of p, q and r negative, and in that case the monodromy is not right-veering. Therefore, these open books cannot support the tight contact structures ξ_m by [7]. □

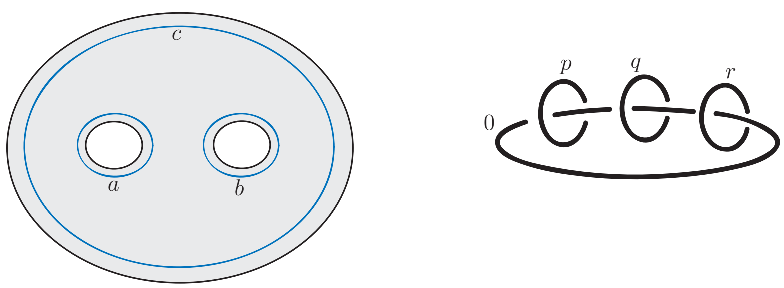

FIGURE 5. A surgery picture of the 3–manifold given by the planar open book with three binding components and monodromy $\phi = \tau_a^p \tau_b^q \tau_c^r$, where τ denotes the right-handed Dehn twist along the corresponding curve

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