

# Heegaard Floer homology of broken fibrations over the circle

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This article is the first in a series where we investigate the relations between Perutz’s Lagrangian matching invariants and Ozsváth-Szabó’s Heegaard Floer invariants of three and four manifolds. In this paper, we deal with the purely Heegaard Floer theoretical side of this programme and prove an isomorphism of 3–manifold invariants for certain  $\text{spin}^c$  structures where the groups involved can be formulated in the language of Heegaard Floer theory. As applications, we give new calculations of Heegaard Floer homology of certain classes of 3–manifolds, a characterization of Juhász’s sutured Floer homology and a proof of Floer’s excision theorem in the context of Heegaard Floer homology.

[57M50](#); [57R17](#)

## 1 Introduction

The results that we present in this paper are formulated in the language of Heegaard Floer homology and interesting by themselves from this perspective. However, the main motivation of our study comes from a different setting, namely that of Lagrangian matching invariants developed by Perutz [23], which are conjecturally isomorphic to Heegaard Floer theoretical invariants. In this paper, we prove an isomorphism between the 3–manifold invariants of these theories for certain  $\text{spin}^c$  structures, namely *quilted Floer homology* and *Heegaard Floer homology*. We also outline how the techniques here can be generalized to obtain an identification of 4–manifold invariants and leave the details to a sequel article.

Before giving a review of both of the above mentioned theories, we give the definition of a broken fibration over  $S^1$ , which will be an important part of the topological setting that we will be working with.

**Definition 1** *A map  $f : Y \rightarrow S^1$  from a closed oriented smooth 3–manifold  $Y$  to  $S^1$  is called a broken fibration if  $f$  is a circle-valued Morse function with all of the critical points having index 1 or 2.*

The terminology is inspired from the terminology of broken Lefschetz fibrations on 4–manifolds, to which we will return later in this paper in Section 4.3. We remark that a 3–manifold admits a broken fibration if and only if  $b_1(Y) > 0$ , and if it admits one, it admits a broken fibration with connected fibres.

We will mostly restrict ourselves to broken fibrations with connected fibres and we will denote by  $\Sigma_{\max}$  and  $\Sigma_{\min}$  two fibres with maximal and minimal genus. We denote by  $\mathcal{S}(Y|\Sigma_{\min})$ , the  $\text{spin}^c$  structures  $\mathfrak{s}$  on  $Y$  such that  $\langle c_1(\mathfrak{s}), [\Sigma_{\min}] \rangle = \chi(\Sigma_{\min})$  (those  $\text{spin}^c$  structures which satisfy the adjunction *equality* with respect to the fibre with minimal genus).

**Definition 2** *The universal Novikov ring  $\Lambda$  over  $\mathbb{Z}$  is the ring of formal power series  $\Lambda = \sum_{r \in \mathbb{R}} a_r t^r$  with  $a_r \in \mathbb{Z}$  such that  $\#\{r | a_r \neq 0, r < N\} < \infty$  for any  $N \in \mathbb{R}$ .*

The main theorem of this paper is an isomorphism, for all  $\text{spin}^c$  structures in  $\mathcal{S}(Y|\Sigma_{\min})$ , between the quilted Floer homology of a broken fibration  $f : Y \rightarrow S^1$  (with coefficients in the universal Novikov ring) and the Heegaard-Floer homology of  $Y$  perturbed by a closed 2-form  $\eta$  that pairs positively with the fibers of  $f$ :

**Theorem 3**  $QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq HF^\pm(Y, \eta, \mathfrak{s})$  for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ .

When  $g(\Sigma_{\min})$  is at least 2 the theorem holds for integral coefficients.

**Corollary 4** *Suppose that  $g(\Sigma_{\min}) > 1$ , then for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$  we have*

$$QFH'(Y, f, \mathfrak{s}; \mathbb{Z}) \simeq HF^+(Y, \mathfrak{s})$$

In Section 2, we construct a Heegaard diagram associated with a broken fibration and investigate the properties of this diagram. We also give a calculation of perturbed Heegaard Floer homology of fibred 3-manifolds for  $\mathfrak{s} \in \mathcal{S}(Y|F)$ . In Section 3, we give a definition of quilted Floer homology in the language of Heegaard Floer theory and prove that it is isomorphic to the Heegaard Floer homology for the  $\text{spin}^c$  structures under consideration. In Section 4, we relate the group defined in Section 3 to the original definition of quilted Floer homology in terms of holomorphic quilts. Here we also prove Floer's excision theorem and discuss the extension of this isomorphism to four-manifold invariants.

We now proceed to review the theories and the notation that are involved in our theorem.

## 1.1 (Perturbed) Heegaard Floer homology

In this section, we review the construction of Heegaard Floer homology, introduced by Ozsváth and Szabó [18]. The usual construction involves certain admissibility conditions, however there is a variant of Heegaard Floer homology where Novikov rings and perturbations by closed 2-forms are introduced in order to make the Heegaard Floer homology group well-defined without any admissibility condition. Our account will be brief since this theory has been well developed in the literature. The reader is encouraged to turn to [5] for a more detailed account of perturbed Heegaard

Floer theory. Furthermore, we will mostly find it convenient to work in the set up of Lipshitz's cylindrical reformulation of Heegaard Floer homology [12].

Let  $(\Sigma_g, \alpha, \beta, z)$  be a pointed Heegaard diagram of a 3-manifold  $Y$ . This gives rise to a pair of Lagrangian tori  $\mathbb{T}_\alpha, \mathbb{T}_\beta$  in  $\text{Sym}^g(\Sigma_g)$ , together with a holomorphic hypersurface  $Z = z \times \text{Sym}^{g-1}(\Sigma_g)$ . The Heegaard-Floer homology of  $Y$  is the Lagrangian Floer homology of these tori, where one uses the orbifold symplectic form pushed down from  $\Sigma_g^{\times g}$ , though one can also use honest symplectic forms (see [22]). The differential is twisted by keeping track of the intersection number  $n_z$  of holomorphic disks contributing to the differential with  $Z$ . More precisely, the Heegaard-Floer chain complex  $CF^+(Y)$  is freely generated over  $\mathbb{Z}$  by  $[\mathbf{x}, i]$  where  $\mathbf{x}$  is an intersection point of  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$  and  $i \in \mathbb{Z}_{\geq 0}$ , and the differential is given by

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi)[\mathbf{y}, i - n_z(\varphi)]$$

The above definition only makes sense under certain admissibility conditions so that the sum on the right hand side of the differential is finite. In general, one can consider a twisted version of the above chain complex by a closed 2-form in  $\Omega^2(Y)$ . This is called the *perturbed Heegaard-Floer homology*. The chain complex  $CF^+(Y, \eta)$  is freely generated over  $\Lambda$  (see Definition 2) by  $[\mathbf{x}, i]$  where  $\mathbf{x}$  is an intersection point and  $i$  is a nonnegative integer as before, and the differential is twisted by the area  $\int_{[\varphi]} \eta$  of the holomorphic disks that contribute to the differential. More precisely, the differential of the perturbed theory is given by

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi) t^{\eta(\varphi)}[\mathbf{y}, i - n_z(\varphi)]$$

Note that if  $\varphi_1, \varphi_2$  are two holomorphic discs that connect an intersection point  $\mathbf{x}$  to  $\mathbf{y}$ , then their difference is a periodic domain  $P$  and we have the equality  $\eta(\varphi_1) - \eta(\varphi_2) = \eta([P])$ , where the latter only depends on the cohomology class of  $\eta$ . We remark that although the differential depends on the choice of a representative of the class  $[\eta]$ , the isomorphism class of the homology groups is determined by  $\text{Ker}(\eta) \cap H_2(Y; \mathbb{Z})$ .

Recall that a 2-form is said to be generic when  $\text{Ker}(\eta) \cap H_2(Y; \mathbb{Z}) = \{0\}$ . For a generic form coming from an area form on the Heegaard surface,  $HF^+(Y, \eta)$  is defined without any admissibility conditions on the Heegaard diagram.

## 1.2 Quilted Floer homology of a 3-manifold

In this section, we review the definition of quilted Floer homology of a 3-manifold  $Y$  equipped with a broken fibration  $f : Y \rightarrow S^1$ . The general theory of holomorphic quilts is under systematic development by Wehrheim and Woodward [28], though the case we consider also appears in the

work of Perutz [21]. The relevant part of the theory in the setting of 3-manifolds is obtained from Perutz’s construction of Lagrangian matching conditions associated with critical values of broken fibrations, which we now review from [23].

Given a Riemann surface  $(\Sigma, j)$  and an embedded circle  $L \subset \Sigma$ , denote by  $\Sigma_L$  the surface obtained from  $\Sigma$  by surgery along  $L$ , i.e., by removing a tubular neighborhood of  $L$  and gluing in a pair of discs. To such data, Perutz associates a distinguished Hamiltonian isotopy-class of Lagrangian correspondences  $V_L \subset \text{Sym}^n(\Sigma) \times \text{Sym}^{n-1}(\Sigma_L)$  (where the symmetric products are equipped with Kähler forms in suitable cohomology classes, see [23]). These are described in terms of a symplectic degeneration of  $\text{Sym}^n(\Sigma)$ . More precisely, one considers an elementary Lefschetz fibration over  $D^2$  with regular fibre  $\Sigma$  and a unique vanishing cycle  $L$  which collapses at the origin. Then one passes to the relative Hilbert scheme,  $\text{Hilb}_{D^2}^n(\Sigma)$ , of this fibration (the resolution of the singular variety obtained by taking fibre-wise symmetric products). The regular fibres of the induced map from  $\text{Hilb}_{D^2}^n(\Sigma)$  are identified with  $\text{Sym}^n(\Sigma)$ , and the fibre above the origin has a codimension 2 singular locus which can be identified with  $\text{Sym}^{n-1}(\Sigma_L)$ .  $V_L$  then arises as the vanishing cycle of this fibration.

Given a 3-manifold  $Y$  and a broken fibration  $f : Y \rightarrow S^1$ , the *quilted Floer homology* of  $Y$ ,  $QFH(Y, f)$ , is a Lagrangian intersection theory graded by  $\text{spin}^c$  structures on  $Y$ . Let  $p_1, p_2, \dots, p_k$  be the set of critical values of  $f$ . Pick points  $p_i^\pm$  in a small neighborhood of each  $p_i$  so that the fibre genus increases from  $p_i^-$  to  $p_i^+$ . For  $\mathfrak{s} \in \text{spin}^c(Y)$ , let  $\nu : S^1 \setminus \text{crit}(f) \rightarrow \mathbb{Z}_{\geq 0}$  be the locally constant function defined by  $\langle c_1(\mathfrak{s}), [F_s] \rangle = 2\nu(s) + \chi(F_s)$ , where  $F_s = f^{-1}(s)$ . Then the construction in the previous paragraph gives Lagrangian correspondences  $L_{p_i} \subset \text{Sym}^{\nu(p_i^+)}(F_{p_i^+}) \times \text{Sym}^{\nu(p_i^-)}(F_{p_i^-})$ . The quilted Floer homology of  $Y$ ,  $QFH(L_{p_1}, \dots, L_{p_k})$ , is then generated by horizontal (with respect to the gradient flow of  $f$ ) multi-sections of  $f$  which match along the Lagrangians  $L_{p_1}, \dots, L_{p_k}$  at the critical values of  $f$ , and the differential counts rigid holomorphic “quilted cylinders” connecting the generators, [21], [28].

There are various technical difficulties involved in the definition of  $QFH(Y, f, s)$  due to bubbling of holomorphic curves. These are addressed by different means depending on the value of  $\langle c_1(\mathfrak{s}), [\Sigma_{\max}] \rangle$ . The easiest case is the monotone case, that is when  $\langle c_1(\mathfrak{s}), [\Sigma_{\max}] \rangle > 0$ , where holomorphic bubbles are a priori excluded. However, for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$  we will almost never be in the monotone case. In the strongly negative case, that is when  $\langle c_1(\mathfrak{s}), [\Sigma_{\max}] \rangle \leq \chi(\Sigma_{\max})/2$ , one can still eliminate bubbles a priori by standard means. For the rest of the cases, bubbles might and will occur in general, therefore complications arise. The main idea is then to establish a proper combinatorial rule for handling bubbled configurations. One could also try to use the more technical machinery of [14] or [4] in order to tackle this case. Another related issue is showing that quilted Floer homology is an invariant of a three manifold. The isomorphism constructed in this paper shows this in an indirect way for the  $\text{spin}^c$  structures under consideration. We will return to this question and various well-definedness questions in [11]. For now we will give an alternative

description which we will denote by  $QFH'(Y, f, s)$  that suits our purposes and avoids these technical issues, hence is well-defined in all cases; see Section 3.1 for the definition, and Section 4 and [11] for the equivalence between the two constructions.

In this paper, we will deal with the  $\text{spin}^c$  structures  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ . In this case, when defined, quilted Floer homology can be interpreted as a variant of the construction of Heegaard Floer homology, because of Theorem 23 (in Section 4) and a version of the main theorem in the work of Wehrheim and Woodward [28]. From now on, we will work with the definition formulated in terms of Heegaard Floer theory as given in Section 3.1.

Finally, we remark that in the case when  $f : Y \rightarrow S^1$  is a fibration,  $QFH(Y, f)$  is given as a fixed point Floer homology theory on the moduli space of vortices and was first introduced by Salamon in [26]. In this case, the  $\text{spin}^c$  structures  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma)$  corresponds to taking the zeroth symmetric product of the fibres. In this case, it is natural to set  $QFH(Y, f) = \Lambda$  if  $\mathfrak{s}$  is the canonical tangent  $\text{spin}^c$  structure, and  $QFH(Y, f) = 0$  for other  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma)$ .

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## 2 Heegaard diagram for a circle-valued broken fibrations on $Y$

### 2.1 A standard Heegaard diagram

We start with a 3-manifold  $Y$  with  $b_1 > 0$ . Then  $Y$  admits a broken fibration over  $S^1$ . Consider such a Morse function  $f : Y \rightarrow S^1$  with the following additional properties :

- $F_{-1} = \Sigma_{\max}$  has the maximal genus  $g_{\max} = g$  and  $F_1 = \Sigma_{\min}$  has the minimal genus  $g_{\min} = k$  among fibres of  $f$ .
- The fibres are connected.
- The genera of the fibres are in decreasing order as one travels clockwise and counter-clockwise from  $-1$  to  $1$ .

A broken fibration with these properties always exists provided  $b_1 > 0$ . In fact, any broken fibration with connected fibers can be deformed into one with these properties by an isotopy that changes the order of the critical values.

We will now construct a Heegaard diagram for  $Y$  adapted to  $f$ . Roughly speaking, the Heegaard surface  $\Sigma$  will be obtained by connecting  $\Sigma_{\max}$  and  $\Sigma_{\min}$  by two “tubes” traveling clockwise and counter-clockwise from  $\Sigma_{\max}$  to  $\Sigma_{\min}$ . More precisely, start with a section  $\gamma$  of  $f$  over  $S^1$ . Then we can pick a metric for which  $\gamma$  is a gradient flow line of  $f$ , and since  $\gamma$  is disjoint from the critical points of  $f$ , it also avoids the stable/unstable manifolds of the critical points. Now pick two distinct points  $p$  and  $q$  on  $\Sigma_{\max}$  sufficiently close to the point where  $\gamma$  intersects  $\Sigma_{\max}$ , connect  $p$  to  $\Sigma_{\min}$  by the gradient flow line above the northern semi-circle in the base  $S^1$  which connects  $-1$  to  $1$  in the clockwise direction and connect  $q$  to  $\Sigma_{\min}$  by the gradient flow line above the southern semi-circle, avoiding the critical points of  $f$  in both cases. Denote these flow lines by  $\gamma_p$  and  $\gamma_q$  and their end points in  $\Sigma_{\min}$  by  $\bar{p}$  and  $\bar{q}$ . Then the Heegaard surface that we are interested in is obtained by removing discs around  $p, q, \bar{p}$  and  $\bar{q}$  and connecting  $\Sigma_{\max}$  to  $\Sigma_{\min}$  along  $\gamma_p$  and  $\gamma_q$  (see Figure 1). We denote the resulting surface by

$$\Sigma = \Sigma_{\max} \cup_{\partial N(\gamma_p) \cup \partial N(\gamma_q)} \Sigma_{\min}$$

where  $N(\gamma_p)$  and  $N(\gamma_q)$  stands for normal neighborhoods of  $\gamma_p$  and  $\gamma_q$ .

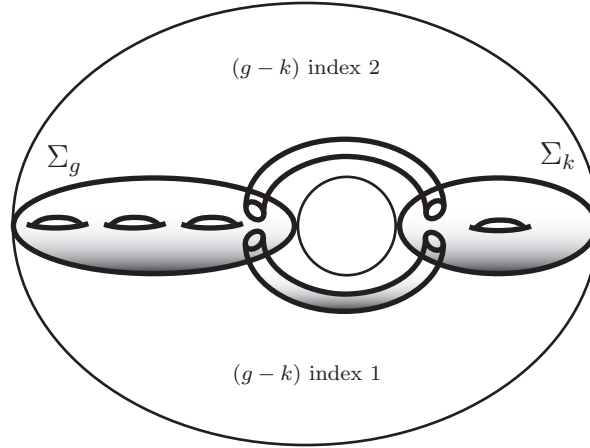


Figure 1: Heegaard surface for a broken fibration

Note that  $g(\Sigma) = g + k + 1$ . Denote the point where  $\gamma$  intersects  $\Sigma_{\max}$  by  $w$  and the point where  $\gamma$  intersects  $\Sigma_{\min}$  by  $z$ . Next, we will describe  $\alpha$  and  $\beta$  curves on  $\Sigma$  in order to get a Heegaard decomposition of  $Y$ . First, set  $\alpha_0$  to be  $\partial N(\gamma_p) \cap f^{-1}(-i)$  and set  $\beta_0$  to be  $\partial N(\gamma_p) \cap f^{-1}(i)$ . The preimage of the northern semi-circle is a cobordism from  $\Sigma_{\max}$  to  $\Sigma_{\min}$  which can be realized by attaching  $(g - k)$  2-handles to  $\Sigma_{\max} \times I$ , and hence can be described by the data of  $g - k$  disjoint attaching circles on  $\Sigma_{\max}$ . These we declare to be  $\alpha_1, \dots, \alpha_{g-k}$ . Similarly the preimage of the southern semi-circle is a cobordism from  $\Sigma_{\max}$  to  $\Sigma_{\min}$ , encoded by  $g - k$  disjoint attaching circles  $\beta_1, \dots, \beta_{g-k}$  on  $\Sigma_{\max}$ . Alternatively, these two sets correspond to the stable and unstable manifolds of the critical points of  $f$ . More precisely, orienting the base  $S^1$  in the clockwise

direction,  $\alpha_1, \dots, \alpha_{g-k}$  are the intersections of the stable manifolds of the critical points above the northern semi-circle with  $\Sigma_{\max}$ , similarly  $\beta_1, \dots, \beta_{g-k}$  are the intersections of the unstable manifolds of the critical points above the southern semi-circle with  $\Sigma_{\max}$ . Note that by choosing  $p$  and  $q$  sufficiently close to  $w$  we can ensure that they lie in the connected component as  $w$  in the complement of  $\alpha_1, \dots, \alpha_{g-k}$  and  $\beta_1, \dots, \beta_{g-k}$ .

Next, we describe the remaining curves,  $(\alpha_{g-k+1}, \dots, \alpha_{g+k}, \beta_{g-k+1}, \dots, \beta_{g+k})$ . Let  $F$  be the part of  $\Sigma$  which consists of  $\Sigma_{\max}$  (except the two discs removed around  $p$  and  $q$ ) together with halves of the connecting tubes up to  $\alpha_0$  and  $\beta_0$ . Thus  $F$  is a genus  $g$  surface with 2 boundary components  $\alpha_0$  and  $\beta_0$ . Also, denote by  $\bar{F}$  the complement of  $\text{Int}(F)$  in  $\Sigma$ . Thus  $\bar{F}$  is a genus  $k$  surface with boundary consisting of  $\alpha_0$  and  $\beta_0$  and  $\Sigma = F \cup_{\alpha_0 \cup \beta_0} \bar{F}$ . Let us also pick  $p^+$  and  $q^+$  on the boundary of the discs deleted around  $p$  and  $q$ , and  $\bar{p}^+$  and  $\bar{q}^+$  their images under the gradient flow (so that they lie on the boundary of the discs deleted around  $\bar{p}$  and  $\bar{q}$ ). Now we can find two  $2k$ -tuples of “standard” pairwise disjoint arcs in  $\bar{F}$ ,  $(\bar{\xi}_1, \dots, \bar{\xi}_{2k}), (\bar{\eta}_1, \dots, \bar{\eta}_{2k})$  such that  $\bar{\xi}_i$  intersect  $\bar{\eta}_j$  only if  $i = j$ , in which case the intersection is transverse at one point. Furthermore, we can arrange that the points  $z, \bar{p}^+$  and  $\bar{q}^+$  lie in the same connected component in the complement of these arcs in  $\bar{F}$ . A nice visualization of these curves on  $\bar{F}$  can be obtained by considering a representation of  $\bar{F}$  by a  $4k$ -sided polygon. First, represent a genus  $k$  surface by gluing the sides of  $4k$ -gon in the way prescribed by the labeling  $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_k b_k a_k^{-1} b_k^{-1}$  of the sides starting from a vertex and labeling in the clockwise direction. Now remove a neighborhood of each vertex of the polygon and a neighborhood of a point in its interior. This now represents a genus  $k$  surface with two boundary components. Let us put  $\beta_0$  at the boundary of the interior puncture and  $\alpha_0$  at the boundary near the vertices then the curves  $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i})$  coincide with the portions of the edges labelled  $(a_i, b_i)$  left after removing a neighborhood of each vertex and the curves  $(\bar{\eta}_{2i-1}, \bar{\eta}_{2i})$  connect the midpoints of  $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i})$  radially to  $\beta_0$ , see Figure 2.

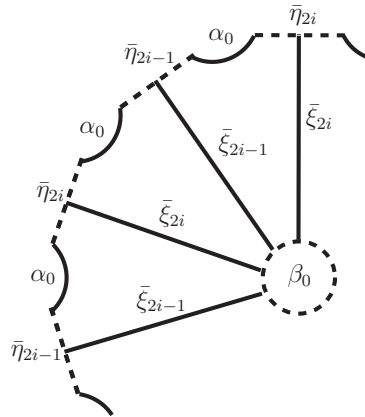


Figure 2: The curves  $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i}), (\bar{\eta}_{2i-1}, \bar{\eta}_{2i})$

Now, using the gradient flow of  $f$  we can flow the arcs  $(\bar{\xi}_1, \dots, \bar{\xi}_{2k})$  above the northern semi-circle to obtain disjoint arcs  $(\xi_1, \dots, \xi_{2k})$  in  $F$  which do not intersect with  $\alpha_1, \dots, \alpha_{g-k}$ . (Generic choices ensure that the gradient flow does not hit any critical points.) The flow sweeps out discs in  $Y$  which bound  $(\alpha_{g-k+1}, \dots, \alpha_{g+k}) = (\xi_1 \cup \bar{\xi}_1, \dots, \xi_{2k} \cup \bar{\xi}_{2k})$ . Similarly, we define  $(\beta_{g-k+1}, \dots, \beta_{g+k})$  by flowing the arcs  $(\bar{\eta}_1, \dots, \bar{\eta}_{2k})$  above the southern semi-circle. To complete the Heegaard decomposition of  $(Y, f)$  we set the base point on  $\bar{F}$  to be  $z$  which lies in the same region as  $\bar{p}^+$  and  $\bar{q}^+$ . Therefore, we constructed a Heegaard decomposition of  $(Y, f)$ . We will also make use of a filtration associated with the base point  $w$  which we can ensure to be located in the same region as  $p^+$  and  $q^+$  by picking  $p$  and  $q$  sufficiently close to  $w$ , which is the image of  $z$  under the gradient flow above the northern and southern semi-circles. Roughly speaking, this point will be used to keep track of the domains passing through the connecting “tubes”.

Note that the Heegaard diagram constructed above might be highly inadmissible. An obvious periodic domain with nonnegative coefficients is given by  $F$ , which represents the fibre class. However, the standard winding techniques will give us a Heegaard diagram where  $F$  (or its multiples) is the *only* potential periodic domain which might prevent our Heegaard diagram from being admissible (which happens if and only if  $k = 1$ ). In fact, we can achieve this by only changing the diagram in the interior of  $F$ , so that the standard configuration of curves on  $\Sigma_{\min}$  is preserved. Furthermore, we will make sure that, in the new Heegaard diagram, the points  $w$ ,  $p^+$  and  $q^+$  remain in the same connected component. To get started, fix an arc  $\delta$  in  $F$ , disjoint from all the  $\alpha$  and  $\beta$  curves and arcs in  $\text{Int}(F)$ , that connects the two boundary components of  $F$  and passes through  $p^+$  and  $q^+$ . We claim that there are  $g + k$  simple closed curves  $\{\gamma_1, \dots, \gamma_{g+k}\}$  in  $F$  such that  $\gamma_i$  do not intersect  $\delta$  and the algebraic intersection of  $\gamma_i$  with  $\alpha_j$  is 1 if  $i = j$  and 0 otherwise (Note that we do not require the curves  $\gamma_1, \dots, \gamma_{g+k}$  to be disjoint). For that, we will show that the curves  $\alpha_1, \dots, \alpha_{g-k}, \xi_1, \dots, \xi_{2k}, \delta$  are linearly independent in  $H_1(F, \partial F)$ . Then the Poincaré-Lefschetz duality implies the existence of the desired simple closed curves in  $F$  which do not intersect  $\delta$ .

**Lemma 5** *The curves  $\alpha_1, \dots, \alpha_{g-k}, \xi_1, \dots, \xi_{2k}, \delta$  are linearly independent in  $H_1(F, \partial F)$ .*

*Proof.* It suffices to show that the complement of  $\alpha_1, \dots, \alpha_{g-k}, \xi_1, \dots, \xi_{2k}, \delta$  in  $F$  is connected. Take any two points  $a, b$  in the complement. Now use the gradient flow along the northern semi-circle to obtain  $\bar{a}$  and  $\bar{b}$ . Also let  $\bar{\delta}$  be the image of  $\delta$  under the flow. Connect  $\bar{a}$  and  $\bar{b}$  in the complement of  $\bar{\xi}_1, \dots, \bar{\xi}_{2k}$  in  $\bar{F}$  with a path that is disjoint from  $\bar{\delta}$  (This is easy because of the standard configuration of curves in  $\bar{F}$ ). Now flow the connecting path back to obtain a path that connects  $a$  and  $b$  in the complement of  $\alpha_1, \dots, \alpha_{g-k}, \xi_1, \dots, \xi_{2k}$ .  $\square$

**Lemma 6** *Given a basis of the abelian group of periodic domains in the form  $F, P_1, \dots, P_n$ , after winding the  $\alpha$  curves sufficiently many times along the curves  $\{\gamma_1, \dots, \gamma_{g+k}\}$ , we can arrange that any periodic domain in the linear span of  $P_i$  has both positive and negative regions on the Heegaard surface. Furthermore, for  $\mathfrak{s} \in \mathcal{S}(Y|_{\Sigma_{\min}})$ , the resulting diagram is weakly admissible if  $k > 1$ .*

*Proof.* This follows by winding successively along the curves  $\{\gamma_1, \dots, \gamma_{g+k}\}$  in  $F$ , first wind along  $\gamma_1$  all the  $\alpha$  curves that intersect  $\gamma_1$ , then wind the resulting curves around  $\gamma_2$ , etc. In this way the  $\alpha$  curves stay disjoint (each winding is actually a diffeomorphism of  $F$  supported near  $\gamma_i$ , and maps disjoint curves/arcs to disjoint curves/arcs). Furthermore, because winding along  $\gamma_i$  is a diffeomorphism of  $F$  isotopic to identity, it preserves the property that  $\alpha_j$  and  $\gamma_k$  have algebraic intersection numbers 1 if  $j = k$ , 0 otherwise. If we had a periodic domain with a nontrivial boundary along  $\alpha_i$ , then after winding sufficiently along  $\gamma_i$ , the multiplicity of some region of the periodic domain with boundary in  $\alpha_i$  becomes negative. The argument for that relies on the observation that, since the total boundary of the periodic domain has algebraic intersection number 0 with  $\gamma_i$ , and since all the other  $\alpha$  curves have algebraic intersection number 0, while  $\alpha_i$  has nonzero algebraic intersection, the boundary of the periodic domain must also include a  $\beta$  curve which has nonzero algebraic intersection number with  $\gamma_i$ . Thus after each winding along  $\gamma_i$ , the domain of the periodic domain which has boundary on  $\alpha_i$  has a region where the multiplicity is decreased. Hence after sufficiently many windings, we can ensure that any periodic domain with boundary in one of  $\alpha_1, \dots, \alpha_{g+k}$  has at least one negative region.

Furthermore, note that a periodic domain is uniquely determined by the part of its boundary which is spanned by  $\{[\alpha_0], \dots, [\alpha_{g+k}]\}$ . Therefore, given a basis  $F, P_1, \dots, P_n$ , after winding sufficiently many times, we can make sure that each  $P_i$  has sufficiently large multiplicities both positive and negative in certain regions of the Heegaard diagram where all other  $P_j$ 's have small multiplicities. Thus for a periodic domain to have only positive multiplicities, it must be of the form  $mF + m_1P_1 + \dots + m_nP_n$  such that  $m$  is much larger than  $|m_i|$ . Then  $\langle c_1(\mathfrak{s}), mF + m_1P_1 + \dots + m_nP_n \rangle = m\langle c_1(\mathfrak{s}), F \rangle + \sum_{i=1}^n m_i\langle c_1(\mathfrak{s}), P_i \rangle$  must be non-zero when  $k \neq 1$  since  $m\langle c_1(\mathfrak{s}), F \rangle$  dominates the sum and  $\langle c_1(\mathfrak{s}), F \rangle = 2 - 2k$  is non-zero. Thus the diagram can be made weakly admissible when  $k > 1$ .  $\square$

We remark that the configuration of the curves on  $\bar{F}$  is left intact. Also, the curve  $\delta$  in  $F$  has not been changed. Therefore, after winding we still have the points  $p$  and  $q$  lying in the same region of the Heegaard diagram. From now on, we will use the notation  $(\Sigma, \alpha_0, \dots, \alpha_{g+k}, \beta_0, \dots, \beta_{g+k}, z, w)$  for this diagram, which is weakly admissible if  $k > 1$ . We will refer to this kind of diagrams as *almost admissible*. In order to make sense of Heegaard Floer homology groups for our special Heegaard diagram in the case when the lowest genus fibre is a torus (i.e.  $k = 1$ ), we will need to work in the perturbed setting since the periodic domain  $F$  prevents the diagram from being weakly admissible. However, because we have an “almost admissible” diagram, it suffices to perturb only in the “direction of the fibre class”.

**Lemma 7** *Given a basis of the abelian group of periodic domains in the form  $F, P_1, \dots, P_n$ , we can find an area form  $A$  on the Heegaard surface such that  $A([F]) > 0$  and  $A(\text{span}\{P_1, \dots, P_n\}) = 0$ .*

*Proof.* By the previous lemma, we can arrange that any periodic domain in the linear span of  $\{P_1, \dots, P_n\}$  has both positive and negative regions on the Heegaard surface. The rest of the proof

now follows from Farkas' lemma in the theory of convex sets. See [13] Lemma 4.17–4.18.  $\square$

Now an area form  $A$  on the Heegaard surface gives a real cohomology class  $[\eta] \in H^2(Y; \mathbb{R})$  via the bijection between periodic domains and  $H_2(Y; \mathbb{Z})$ . Namely, set  $[\eta](P) = A(P)$ . Choosing a representative  $\eta \in [A]$  we can consider the perturbed Heegaard Floer homology  $HF^+(Y, f, \eta)$ . Since  $F$  is the only periodic domain which prevents weak admissibility (only in the case  $k = 1$ ) and  $\eta([F]) > 0$ , we have a well-defined group  $HF^+(Y, f, \eta)$  by the following lemma :

**Lemma 8** *Given  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i, j \in \mathbb{Z}_{\geq 0}$  and  $r, s \in \mathbb{R}$  there are only finitely many homology classes  $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ , with  $n_z(\varphi) = i - j$  and  $\eta(\varphi) = r - s$  which have positive domains.*

*Proof.* Let  $\varphi$  and  $\psi$  be in  $\pi_2(\mathbf{x}, \mathbf{y})$ , then  $\varphi - \psi \in \pi_2(\mathbf{x}, \mathbf{x})$ . We can write  $\varphi - \psi = mF + m_1P_1 + \dots + m_nP_n + n\Sigma$ . Since  $n_z(\varphi) = n_z(\psi)$ , we have  $n = 0$ . Also since  $\eta(\varphi) = \eta(\psi)$  and  $\eta(F) \neq 0$  while  $\eta(P_i) = 0$ , we conclude that  $m = 0$ . Finally, since  $A(P_i) = 0$ , we have  $A(\varphi) = A(\psi)$  but then there are only finitely many nonnegative domains which have a fixed area.  $\square$

Now, as explained in the introduction  $HF^+(Y, f, \eta)$  is an invariant of  $(Y, [\eta])$ , in fact it only depends on  $\text{Ker}(\eta) \cap H_2(Y; \mathbb{Z})$ , hence is independent of the value of  $\eta([F])$ .

The usual invariance arguments of Heegaard Floer theory, as in [18], imply that  $HF^+(Y, f, \eta)$  is independent of the choice of  $f$  within its smooth isotopy class. Also note that a geometric way of choosing  $\eta$  is by choosing a section  $\gamma$  of  $f$  (a section of  $f$  always exists) and letting  $[\eta]$  be the Poincaré dual of  $[\gamma]$ . In that case, we will write  $HF^+(Y, f, \gamma)$  for this perturbed Heegaard Floer homology group. In fact, the choice of the base points  $w$  and  $z$  as above gives a section of  $f$ . Namely, note that we have arranged so that the image of  $z$  under the flow above both the northern and the southern semi-circles lies in the same region as  $w$ . The union of these two gradient flow lines can therefore be perturbed into a section of  $f$ , which we will denote by  $\gamma_w$ . The group  $HF^+(Y, f, \gamma_w)$  will be one of the main protagonists in this paper. The differential of this group can be made more explicit as follows: Choose a basis of the group of periodic domains in the form  $F, P_1, \dots, P_n$  such that  $F$  is the fibre of  $f$  and  $P_i$  are periodic domains so that the boundary of  $P_i$  does not include  $\alpha_0$  or  $\beta_0$  (This can be arranged by subtracting a multiple of  $F$ ). Then if we choose  $\eta \in PD[\gamma_w]$  we have  $\eta(\text{span}(P_1, \dots, P_m)) = 0$  and  $\eta(F) = n_w(F) = 1$ . Therefore for any periodic domain  $P$ , we have  $\eta(P) = n_w(P)$ . Thus there exists a function  $\lambda : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rightarrow \mathbb{R}$  such that for any  $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$ , we have  $\eta(\varphi) - n_w(\varphi) = \lambda(\mathbf{x}) - \lambda(\mathbf{y})$ . Hence, we can define the differential for  $HF^+(Y, f, \gamma_w)$  as follows:

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi) t^{n_w(\varphi)} [\mathbf{y}, i - n_z(\varphi)]$$

This yields the same homology groups as the original definition where the differential is weighted by  $t^{\eta(\varphi)}$ : namely, the two chain complexes are related by rescaling each generator  $[\mathbf{x}, i]$  to  $t^{\lambda(\mathbf{x})}[\mathbf{x}, i]$ .

When we consider  $HF^+(Y, f, \gamma_w)$ , we will always consider the differential above.

## 2.2 Splitting the Heegaard diagram

As explained in the introduction, we will only consider the  $\text{spin}^c$  structures on  $Y$  that satisfy the adjunction equality with respect to  $\Sigma_{\min}$ ; the set of isomorphism classes of such  $\text{spin}^c$  structures was denoted by  $\mathcal{S}(Y|\Sigma_{\min})$ . In this section we observe that for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ , we obtain a nice splitting of the generators of the Heegaard Floer complex into intersections in  $F$  and  $\bar{F}$ . Furthermore, we prove a key lemma en route to understanding the holomorphic curves contributing to the differential.

Let us denote by  $I_{\text{left}}$  the intersection of  $\alpha_1 \times \dots \times \alpha_{g-k}$  and  $\beta_1 \times \dots \times \beta_{g-k}$  in  $\text{Sym}^{g-k}(\Sigma)$ , and by  $I_{\text{right}}$  the set of intersection points of  $\alpha_0 \times \alpha_{g-k+1} \times \dots \times \alpha_{g+k}$  and  $\beta_0 \times \beta_{g-k+1} \times \dots \times \beta_{g+k}$  in  $\text{Sym}^{2k+1}(\Sigma)$  such that each intersection point *lies* in  $\bar{F}$ . Thus, each element of  $I_{\text{right}}$  consists of one point from the set of  $4k$  intersection points of  $\alpha_0$  with  $\eta_1, \dots, \eta_{2k}$ , another point from the set of  $4k$  intersection points of  $\beta_0$  with  $\xi_1, \dots, \xi_{2k}$  and finally  $2k - 1$  points from the set of  $2k$  points consisting of the intersections of  $\bar{\xi}_i$  with  $\bar{\eta}_i$  for  $i = 1, \dots, 2l$ .

We have  $I_{\text{left}} \otimes I_{\text{right}} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ , where  $\mathbb{T}_\alpha = \alpha_0 \times \dots \times \alpha_{g+k}$  and  $\mathbb{T}_\beta = \beta_0 \times \dots \times \beta_{g+k}$  are the Heegaard tori in  $\text{Sym}^{g+k+1}(\Sigma)$ . Denote by  $C_{\text{left}}$  and  $C_{\text{right}}$  the free  $\Lambda$ -modules generated by  $I_{\text{left}}$  and  $I_{\text{right}}$  respectively.

**Lemma 9** *An intersection point  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  induces a  $\text{spin}^c$  structure  $\mathfrak{s}_z(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$  if and only if  $\mathbf{x} \in C_{\text{left}} \otimes C_{\text{right}}$ .*

*Proof.* This follows easily from the following formula from Lemma 4.11 in [12]:

$$\langle c_1(\mathfrak{s}_z(\mathbf{x})), F \rangle = e(F) + 2n_{\mathbf{x}}(F)$$

where  $n_{\mathbf{x}}(F)$  is the number of components of the tuple  $\mathbf{x}$  which lie in  $F$ . Since  $\mathfrak{s}_z(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$ , we have  $\langle c_1(\mathfrak{s}_z(\mathbf{x})), F \rangle = \langle c_1(\mathfrak{s}_z(\mathbf{x})), \Sigma_{\min} \rangle = 2 - 2k$ . Also  $e(F) = -2g$ , hence the above formula gives

$$n_{\mathbf{x}}(F) = 1 + g - k$$

which is satisfied if and only if  $\mathbf{x} \in C_{\text{left}} \otimes C_{\text{right}}$ .  $\square$

Next, we prove an important lemma about the behaviour of holomorphic disks on the tubular regions to the left of  $\alpha_0$  and  $\beta_0$ . This lemma lies at the heart of most of the arguments about the behaviour of holomorphic curves that we are going to consider subsequently. For the purpose of stating the next lemma, let  $a$  and  $b$  be parallel pushoffs of  $\alpha_0$  and  $\beta_0$  to the left (into the interior of  $F$ ). Let us label the connected components of the domains in the cylindrical region between  $a$  and  $\alpha_0$  by  $a_1, \dots, a_{4k}$  and the cylindrical region between  $b$  and  $\beta_0$  by  $b_1, \dots, b_{4k}$ . Choose the labeling so that  $a_1$  and  $b_1$  are in the same region as the arc  $\delta$ , hence  $n_{a_1} = n_{b_1} = n_w$ .

**Lemma 10** *Let  $\mathbf{x} = \mathbf{x}_{\text{left}} \otimes \mathbf{x}_{\text{right}}$  and  $\mathbf{y} = \mathbf{y}_{\text{left}} \otimes \mathbf{y}_{\text{right}}$  be in  $C_{\text{left}} \otimes C_{\text{right}}$  and  $A \in \pi_2(\mathbf{x}, \mathbf{y})$  and  $u$  be a Maslov index 1 holomorphic curve in the homology class  $A$ . Assume moreover that the contribution of curves in class  $A$  to the differential is non-zero. Then,*

$$n_w(u) = n_{a_1}(u) = \dots = n_{a_{4k}}(u) = n_{b_1}(u) = \dots = n_{b_{4k}}(u)$$

*Furthermore, if  $n_z(u) = 0$ , then the projection to the Heegaard surface induced by  $u$  can be arranged to be an unbranched cover around the cylindrical neighborhoods of  $a$  and  $b$  (In other words,  $u$  “converges” to Reeb orbits around  $a$  and  $b$  upon neck-stretching).*

*Proof.* The proof will be obtained by “stretching the neck” along the curves  $a$  and  $b'$ , where  $a$  is as before a parallel pushoff of  $\alpha_0$  to the left, whereas  $b'$  is a parallel pushoff of  $\beta_0$  to the right (into the interior of  $\bar{F}$ ), chosen so that the marked point  $z$  lies in between  $\beta_0$  and  $b'$ . We could just as well do the stretching along  $a$  and  $b$  and get the first part of the statement, however it turns out that stretching the neck around  $a$  and  $b'$  (and symmetrically  $a'$  and  $b$ , where  $a'$  is similarly a parallel pushoff of  $\alpha_0$  to the right) gives the stronger result stated above.

Suppose that there is an  $i \pmod{4k}$  such that  $n_{a_i}(u) \neq n_{a_{i+1}}(u)$  (one can argue in the same way for  $b_i$ 's). Thus the source  $S$  of  $u$  has a piece of boundary which maps to the  $\beta$  arc that separates  $a_i$  and  $a_{i+1}$ . Let  $\beta_j$  be the curve containing that arc. The crucial observation is that the disk  $u$  has no corners in  $\beta_j \cap F$ , since  $\mathbf{x}$  and  $\mathbf{y}$  have no components in  $\beta_j \cap F$ .

We now degenerate  $\Sigma$  along the curves  $a$  and  $b'$ . Specifically, this means that one takes small cylindrical neighborhoods of the curves  $a$  and  $b'$ , and changes the complex structure in that neighborhood so that the modulus of the cylindrical neighborhoods gets larger and larger. Topologically this degeneration can be understood as follows: After degenerating along  $a$  and  $b'$ ,  $\Sigma$  degenerates into  $\Sigma_{\text{max}}$  and  $\Sigma_{\text{min}}$  and the homology class  $A$  splits into  $A_{\text{left}}$  and  $A_{\text{right}}$  corresponding to the induced domains on  $\Sigma_{\text{max}}$  and  $\Sigma_{\text{min}}$  from the domain of  $A$  on  $\Sigma$ . (The definition of homology classes  $\pi_2(\mathbf{x}, \mathbf{y})$  in this degenerated setting is given in Definition 4.8 of [13], it is the homology classes of maps to  $\Sigma_{\text{max}} \times [0, 1] \times \mathbb{R}$  (and to  $\Sigma_{\text{min}} \times [0, 1] \times \mathbb{R}$ ) which have strip-like ends converging to  $\mathbf{x}$  and  $\mathbf{y}$ , and to Reeb chords at points of degeneration).

Next we analyze the holomorphic degeneration of  $u$ . Suppose that the moduli space of holomorphic curves representing  $A$  is non-empty for all large values of the stretching parameter. Then we conclude by Gromov compactness that there is a subsequence converging to a pair of holomorphic combs of height 1 (in the sense of [13] section 5.3, see proposition 5.20 for the proof of Gromov compactness in this setting)  $u_0$  representing  $A_{\text{left}}$  and  $u_1$  representing  $A_{\text{right}}$  (the limiting curves have height 1 because otherwise one of the stages would have index  $\leq 0$ , contradicting transversality – see Proposition 5.5 of [13]). By assumption, the degeneration of  $u$  involves breaking along a Reeb chord  $\rho$  contained in  $a$  with one of the ends on  $a \cap \beta_j$ . Hence some component  $S_0$  of the domain of  $u_0$  has a boundary component  $\Gamma$ , consisting of arc components separated by boundary marked points, such that one of the arcs is mapping to  $\beta_j$  and, at one end of that arc,  $u_0$  has a

strip-like end converging to the Reeb chord  $\rho$ . Now, since there are no corner points on any of the  $\beta$ -arcs in  $\Sigma_{\max}$ , the marked points on  $\Gamma$  are all labeled by Reeb chords on  $a$  (corresponding to arcs connecting intersection points of  $\beta$  curves with  $a$ ), and any two consecutive punctures on  $\Gamma$  are connected by an arc which is mapped to part of a  $\beta$  arc which lies on the left half of the Heegaard diagram. Thus, in particular there are no arcs in  $\Gamma$  which map to  $\alpha$  curves. Now, we can extend  $u_0$  at the punctures on  $\Gamma$  by sending the marked points to the point of  $\Sigma_{\max}$  to which  $a$  collapses upon neck-stretching (This is possible since, after collapsing  $a$ ,  $u_0|_{S_0}$  viewed as a map to  $\Sigma_{\max}$  admits a continuous extension at these points. Note that the projection to  $[0, 1] \times \mathbb{R}$  also extends continuously at the punctures by the definition of holomorphic combs, see the proof of proposition 5.20 [13] for more details regarding this). Therefore, the image of the boundary component  $\Gamma$  under the projection to  $[0, 1] \times \mathbb{R}$  remains bounded and is entirely contained in  $0 \times \mathbb{R}$ . Moreover, since the projection is holomorphic, the projection of  $\Gamma$  to  $0 \times \mathbb{R}$  is a non-increasing function, and hence we conclude that  $\Gamma$  maps to a constant. Now, the maximum principle implies that the entire component  $S_0$  has to be mapped to a constant value in  $0 \times \mathbb{R}$ . Therefore,  $S_0$  has all of its boundary components mapped to  $\beta$  curves. Furthermore, the image of  $u_0$  restricted to boundary of  $S_0$  does not intersect  $\beta_0$  because, even after the degeneration,  $\beta_0$  does not intersect any other  $\beta$  curves; there cannot be a boundary component which is entirely mapped to  $\beta_0$ , since those type of boundaries are not allowed in the Heegaard Floer differential, and the behaviours of  $u$  and  $u_0$  are the same around  $\beta_0$  as the degeneration takes place outside of a neighborhood of  $\beta_0$ . Therefore,  $u_0$  maps all of its boundary to  $\beta$  curves other than  $\beta_0$  in  $\Sigma_{\max}$ . Thus,  $u_0$  restricted to  $S_0$  gives a homological relation between those  $\beta$  curves. However the  $\beta$  curves other than  $\beta_0$  remain linearly independent in homology, even after the degeneration (some of them intersect at the degeneration point). Hence, the chain represented by  $u_0(S_0)$  has to be a multiple of  $[\Sigma_{\max}]$ , which contradicts the assumption that  $n_{a_i}(u_0|_{S_0}) \neq n_{a_{i+1}}(u_0|_{S_0})$  and thus proves the first part of the lemma.

Furthermore, suppose  $n_z = 0$ , and after stretching the neck around  $a$ , suppose that  $u$  is not an unbranched cover around  $a$ , which means that  $u_0$  has to have at least one component  $S_0$  which has a boundary marked point where  $u_0$  converges to a Reeb chord around  $a$ . The argument above then gives that  $u_0(S_0)$  has to be a multiple of  $[\Sigma_{\max}]$ . However the marked point  $z$  lies in one of the domains in between  $b'$  and  $\beta_0$ , hence  $n_z(u) = n_z(u_0) = 0$  does not allow  $u_0$  to surject onto  $\Sigma_{\max}$ . Thus we arrive at a contradiction, which gives the second part of the lemma.  $\square$

### 2.3 Calculations for fibred 3-manifolds and $C_{\text{right}}$

Before delving into a general study of Heegaard Floer homology for broken maps, here we will calculate  $HF^+(Y, \eta)$  in the case of fibred 3-manifolds. Some of these calculations were done independently by Wu in [29], where perturbed Heegaard Floer homology for  $\Sigma_g \times S^1$  is calculated for all  $\text{spin}^c$  structures. We take the liberty to reconstruct some of the arguments presented there in this section since these calculations will play a role for the calculations we do for general

fibred 3–manifolds. Even though we will do calculations in general for any fibred 3–manifold, we will restrict ourselves to  $\text{spin}^c$  structures in  $\mathcal{S}(Y|F)$ , which will simplify the calculations. Our conclusion is that  $\bigoplus_{\mathfrak{s} \in \mathcal{S}(Y|F)} HF^+(Y, \eta)$  has rank 1. See also [1] for a different approach in the case of torus bundles.

For fibred 3–manifolds, we have  $g = k$ , thus the Heegaard diagram has the curves  $\alpha_0, \beta_0$ , and the rest of the diagram is constructed from the standard configuration of curves  $\bar{\xi}_1, \dots, \bar{\xi}_{2k}, \bar{\eta}_1, \dots, \bar{\eta}_{2k}$  as in Figure 2. Also we will see below that, for the  $\text{spin}^c$  structures in  $\mathcal{S}(Y|F)$ , the generators of our chain complex are given by the intersection points in  $C_{\text{right}}$ .

We first discuss the case of torus bundles. It will then be clear that the general case is just a matter of notational complication. Also note that, in the case of torus bundles, we have to use a perturbation  $\eta$  with  $\eta([F]) > 0$  as explained in the previous section since our diagram is not weakly admissible. For higher genus fibrations, the diagram is weakly admissible hence our calculation also determines the unperturbed Heegaard Floer homology  $HF^+(Y)$ . When doing explicit calculations we will always consider the case of  $HF^+(Y, f, \gamma_w)$  but clearly all arguments go through for any perturbation with  $\eta$  satisfying  $\eta([F]) > 0$ , or for the unperturbed case whenever the diagram is weakly admissible.

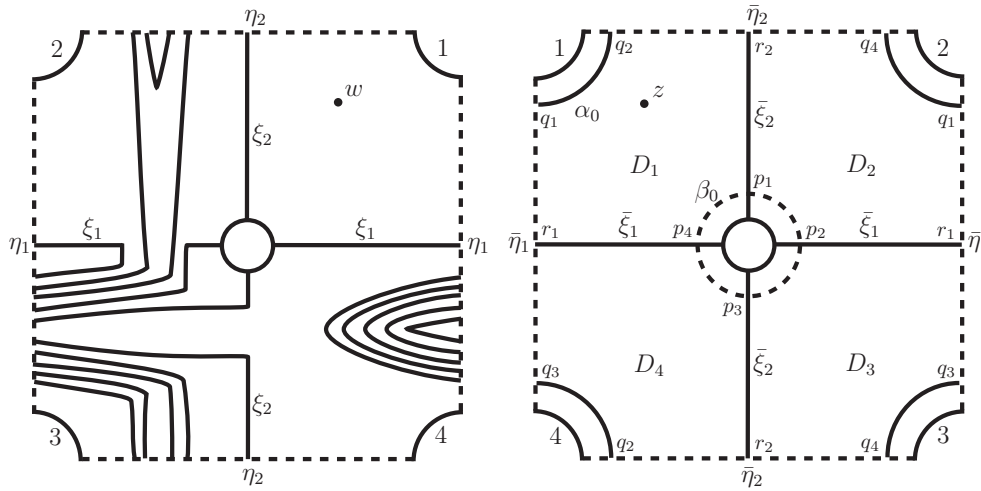


Figure 3: Torus bundles

Figure 3 shows the Heegaard diagram for  $T^3$ . Both the left and the right figure are twice punctured tori, and are identified along the two boundaries (the one in the middle and the one formed by the four corners) where the gluing of the left and right figures is made precise by the labels at the four corners. On the right side the standard set of arcs  $\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2$  are depicted; the left side is constructed by taking the images of these arcs under the horizontal flow (which is the identity map for  $T^3$ ), and winding  $\xi_1$  and  $\xi_2$  along transverse circles so that the diagram becomes almost

admissible (Note that the winding process avoids the region where  $w$  is placed, as required: first  $\xi_2$  is wound once along a horizontal circle, then  $\xi_1$  is wound twice along a vertical circle). For general torus bundles, the same construction will give a Heegaard diagram, where  $\xi_1$  and  $\xi_2$  are replaced by their images under the monodromy of the torus bundle. The important observation here is that the right side of the diagram is always standard. We will show that all the calculations that we need can be done on the right side of the diagram for the  $\text{spin}^c$  structures we have in mind. The calculation for  $T^3$  is essentially the same as in [29]. However, we will see that Lemma 10 plays a crucial role in the calculation for general torus bundles. We first do the calculation for  $T^3$ .

**Proposition 11**  $HF^+(T^3, f, \gamma_w, \mathfrak{s}_0) = \Lambda$  where  $\mathfrak{s}_0 \in \mathcal{S}(T^3|T^2)$  is the unique torsion  $\text{spin}^c$  structure on  $T^3$ .

*Proof.* As in Lemma 9,  $\mathfrak{s}_{\mathbf{x}}(z) \in \mathcal{S}(T^3|T^2)$  if and only if  $\mathbf{x} \in C_{\text{right}}$ , hence  $\mathbf{x}$  can be one of the following tuples of intersections depicted in Figure 3:

$$\begin{aligned} \mathbf{x}_1 &= p_1q_2r_1 & \mathbf{x}_2 &= p_2q_1r_2 & \mathbf{x}_3 &= p_3q_4r_1 & \mathbf{x}_4 &= p_4q_3r_2 \\ \mathbf{y}_1 &= p_4q_1r_2 & \mathbf{y}_2 &= p_1q_4r_1 & \mathbf{y}_3 &= p_2q_3r_2 & \mathbf{y}_4 &= p_3q_2r_1 \end{aligned}$$

Next, we apply the adjunction inequality for the other  $T^2$  components, this implies that the Heegaard Floer groups vanish except for the unique torsion  $\text{spin}^c$  structure,  $\mathfrak{s}_0$  which has  $c_1(\mathfrak{s}_0) = 0$ . The two other torus components are realized by periodic domains in Figure 3, one of them is the domain  $P_1$  including  $D_2 \cup D_3$  and bounded by  $\alpha_2$  and  $\beta_1$ , the other one is the domain  $P_2$  including  $D_3 \cup D_4$  and bounded by  $\alpha_1$  and  $\beta_2$ . Then the formula  $\langle c_1(\mathfrak{s}_z(\mathbf{x})), P_i \rangle = e(P_i) + 2n_{\mathbf{x}}(P_i)$ , implies that the only intersection points for which  $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_0$  are  $\mathbf{x}_1$  and  $\mathbf{y}_1$ . Furthermore, note that  $D_1$  is a hexagonal region connecting  $\mathbf{x}_1$  to  $\mathbf{y}_1$ , hence it is represented by a holomorphic disk  $\varphi_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$ , and the algebraic number of holomorphic disks in the corresponding moduli space of disks in the homology class of  $\varphi_1$  is given by  $\#\widehat{\mathcal{M}}(\varphi_1) = \pm 1$  (See appendix in [25]).

Now, given any other Maslov index 1 homology class  $A \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$ , we have  $A = D_1 + mF + m_1P_1 + m_2P_2$ . In particular, note that  $n_z(A) = 1$ . Furthermore, if we restrict to those with  $n_w = 0$  (that is  $m = 0$ ), since  $m_1P_1 + m_2P_2$  has both positive and negative domains by almost admissibility, unless  $m_1 = m_2 = 0$  there is no holomorphic representative of  $A$ .

We conclude that  $\partial^+[\mathbf{x}_1, i] = f(t)[\mathbf{y}_1, i - 1]$ , where  $f(t) = \pm 1 + O(t)$  is invertible in the Novikov ring. This implies that  $[\mathbf{y}_1, i]$  is in the image of  $\partial^+$ . Thus in particular we have  $\partial^+[\mathbf{y}_1, i] = 0$  for all  $i$ . Finally, there is no Maslov index 1 disk with  $n_w = 0$  which connects  $\mathbf{x}_1$  to itself or  $\mathbf{y}_1$  to itself. Thus we conclude that in  $CF^+(T^3, f, \gamma_w, \mathfrak{s}_0)$ :

$$\begin{aligned} \partial^+[\mathbf{x}_1, 0] &= 0 & \partial^+[\mathbf{y}_1, i] &= 0 \\ \partial^+[\mathbf{x}_1, i] &= (\pm 1 + O(t))[\mathbf{y}_1, i - 1] & \text{for } i > 0 \end{aligned}$$

Hence the homology is generated by  $[\mathbf{x}_1, 0]$ , in other words  $HF^+(T^3, f, \gamma_w, \mathfrak{s}_0) = \Lambda$  as required.

□

From now on, we will simply write  $\mathbf{x}_1$  for  $[\mathbf{x}_1, 0]$ . The next theorem generalizes this calculation to any torus bundle.

**Theorem 12** *Let  $(Y, f)$  be a torus bundle and let  $\mathfrak{s}$  be in  $\mathcal{S}(Y|T^2)$ . Then,  $HF^+(Y, f, \gamma_w, \mathfrak{s}) = \Lambda$  if  $\mathfrak{s} = \mathfrak{s}_0$  where  $\mathfrak{s}_0$  is the  $\text{spin}^c$  structure corresponding to vertical tangent bundle and  $HF^+(Y, f, \gamma_w, \mathfrak{s}) = 0$  otherwise.*

*Proof.* The main difficulty for the general torus bundle case that makes the calculation different from the calculation for  $T^3$  is that we cannot a priori eliminate the generators  $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$  and  $\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ . In fact, if the first Betti number of the torus bundle is equal to 1, these generators are in the same  $\text{spin}^c$  class as  $\mathbf{x}_1$  and  $\mathbf{y}_1$ .

Now, the domains  $D_i$  are homology classes in  $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$ , which have holomorphic representatives  $\varphi_i$  with  $\#\mathcal{M}(\varphi_i) = \pm 1$ . Since any non-trivial periodic domain has to pass through some region to the left of  $\alpha_0$  or  $\beta_0$ , any other homology class in  $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$  which contributes to the differential has to have  $n_w \neq 0$  by Lemma 10. For the same reason, any homology class in  $\pi_2(\mathbf{x}_i, \mathbf{y}_j)$  for some  $i \neq j$  which contributes to the differential has to have  $n_w \neq 0$  since there is no homology class in  $\pi_2(\mathbf{x}_i, \mathbf{y}_j)$  that lies in the right side of the diagram (this can be verified either by inspection, or referring to the case of  $T^3$ , where  $\mathbf{x}_i$  and  $\mathbf{y}_j$  represent different  $\text{spin}^c$  classes for  $i \neq j$ ). Moreover, the classes in  $\pi_2(\mathbf{x}_i, \mathbf{x}_j)$  all have even Maslov index, hence do not contribute to the differential. Therefore, we have

$$\begin{aligned} \partial^+[\mathbf{x}_1, i] &= [\mathbf{y}_1, i - 1] \pmod{t} \text{ for } i > 0 \\ \partial^+[\mathbf{x}_1, 0] &= 0 \pmod{t} \quad \partial^+[\mathbf{x}_2, i] = [\mathbf{y}_2, i] \pmod{t} \\ \partial^+[\mathbf{x}_3, i] &= [\mathbf{y}_3, i] \pmod{t} \quad \partial^+[\mathbf{x}_4, i] = [\mathbf{y}_4, i] \pmod{t} \end{aligned}$$

where the higher order terms do not involve the  $\mathbf{x}_j$ 's. As before, because we are working over a Novikov ring of power series, we conclude that  $[\mathbf{y}_1, i], [\mathbf{y}_2, i], [\mathbf{y}_3, i]$  and  $[\mathbf{y}_4, i]$  are all in the image of  $\partial^+$ . Furthermore, the only possible generator which might be in the kernel of  $\partial^+$  is  $[\mathbf{x}_1, 0]$ . Finally lemma 13 below shows that there is no holomorphic disk starting at  $\mathbf{x}_1$  with  $n_z = 0$  and  $n_w \neq 0$ . Hence we have  $\partial^+[\mathbf{x}_1, 0] = 0$  and the homology group  $\bigoplus_{\mathfrak{s} \in \mathcal{S}(Y|T^2)} HF^+(Y, f, \gamma_w, \mathfrak{s})$  is generated by  $[\mathbf{x}_1, 0]$ . Furthermore,  $\mathfrak{s}_z(\mathbf{x}_1) = \mathfrak{s}_0$  so the theorem is proved. □

Note also that the adjunction inequality implies that  $HF^+(Y, f, \gamma_w, \mathfrak{s})$  vanishes for  $\mathfrak{s} \notin \mathcal{S}(Y|T^2)$ . Therefore the above calculation is in fact a complete calculation of perturbed Heegaard Floer homology for torus bundles.

The following lemma which we alluded to in the above calculation holds in general (not only in the fibred case). Let  $Y$  be any 3-manifold with  $b_1 > 0$ , and  $f : Y \rightarrow S^1$  a broken fibration with connected fibres. Construct the almost admissible Heegaard diagram for  $f$  as before and let

$\mathbf{x}_1 \in C_{\text{right}}$  be given by the union of the intersection points in  $\alpha_0 \cap \beta_2$ ,  $\alpha_2 \cap \beta_0$ , and  $\bar{\xi}_i \cap \bar{\eta}_i$  for  $i \neq 2$ , where the intersection point in  $\alpha_0 \cap \beta_2$  and  $\alpha_2 \cap \beta_0$  are chosen so that the region containing  $z$  includes them as corners. (In the case of the torus bundle this is the generator  $[\mathbf{x}_1, 0]$ ). Note that the generators of  $C_{\text{right}}$  can always be described from the standard diagram since the right hand side of our Heegaard diagrams is always the same.

**Lemma 13** *Let  $\varphi \in \pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{y}_{\text{right}})$  be a holomorphic disk in a class that contributes non-trivially to the differential for given  $\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}}, \mathbf{y}_{\text{right}}$ . If  $n_z(\varphi) = 0$ , then  $\mathbf{y}_{\text{right}} = \mathbf{x}_1$  and the domain of  $\varphi$  is contained on the left side of the Heegaard diagram (i.e. it is contained in  $F$ ).*

*Proof.* Consider the component of  $\mathbf{x}_1$  which is an intersection point on  $\beta_0$ , say  $p_1$ . Now, among the four regions which have  $p_1$  as one of their corners, one includes  $z$ , namely  $D_1$ , and two of them lie in the left half of the diagram, hence by lemma 10, they must have the same multiplicity. Denote these regions by  $L_1$  and  $L_2$ , so that  $L_1$  and  $D_1$  share an edge on  $\beta_0$ . If the component of  $\varphi$  which is asymptotic to  $p_1$  is constant, then  $p_1$  is also part of  $\mathbf{y}_{\text{right}}$ . Otherwise, since  $\varphi$  has a corner which leaves  $p_1$  and  $n_z(\varphi) = 0$ , we must have a non-zero multiplicity at  $L_2$ , but since  $L_1$  and  $L_2$  must have the same multiplicity, this implies that  $p_1$  has to be a member in  $\mathbf{y}_{\text{right}}$ . The same conclusion applies for the point of  $\mathbf{x}_1$  which lies on  $\alpha_0$ . But then there is a unique way to complete these two intersection points to a generator in  $C_{\text{right}}$ , hence we conclude that  $\mathbf{y}_{\text{right}} = \mathbf{x}_1$ . Thus  $\varphi$  is in  $\pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$ .

Furthermore, since  $\varphi$  fixes  $\mathbf{x}_1$ , the intersection of the domain of  $\varphi$  with  $\bar{F}$  must coincide with the intersection of some periodic domain for  $S^1 \times \Sigma_k$  with  $\bar{F}$  (since any domain that has no corners on the right side, can be completed to a periodic domain on the Heegaard diagram of  $S^1 \times \Sigma_k$  by reflecting). However, it is easy to identify all the periodic domains of  $S^1 \times \Sigma_k$  and observe that no non-trivial combination of periodic domains for  $S^1 \times \Sigma_k$  (if we leave out  $F$  and its multiples), can have the same multiplicity in the regions immediately to the left of  $\alpha_0$  and  $\beta_0$ . However, by Lemma 10 this property has to hold. This proves the lemma.  $\square$

**Theorem 14** *Let  $(Y, f)$  be a fibre bundle with fibre a genus  $g$  surface and let  $\mathfrak{s}$  be in  $S(Y|\Sigma_g)$ . Then,  $HF^+(Y, f, \gamma_w, \mathfrak{s}) = \Lambda$  if  $\mathfrak{s} = \mathfrak{s}_0$  where  $\mathfrak{s}_0$  is the  $\text{spin}^c$  structure corresponding to vertical tangent bundle and  $HF^+(Y, f, \gamma_w, \mathfrak{s}) = 0$  otherwise.*

*Proof.* The proof is essentially the same as the proof of the corresponding theorem for the torus bundles. The only difference is the number of generators which are cancelled out: there are now  $8g$  generators  $\mathbf{x}_1, \dots, \mathbf{x}_{4g}$ ,  $\mathbf{y}_1, \dots, \mathbf{y}_{4g}$ , and the  $4g$  hexagonal regions of  $\bar{F}$  (see Figure 2) give  $\partial^+[\mathbf{x}_1, i] = [\mathbf{y}_1, i - 1] \pmod{t}$  and  $\partial^+[\mathbf{x}_j, i] = [\mathbf{y}_j, i] \pmod{t}$  for  $j \geq 2$ . Arguing as before, the only generator left is again  $\mathbf{x}_1$  which gives  $\mathfrak{s}_z(\mathbf{x}_1) = \mathfrak{s}_0$ .  $\square$

Note that this gives a new way of obtaining the results of the original calculation of Ozsváth and Szabó in [18] for fibred 3-manifolds.

**Corollary 15** *Let  $(Y, f)$  be a fibre bundle with fibre a genus  $g > 1$  surface and let  $\mathfrak{s}$  be in  $\mathcal{S}(Y|\Sigma_g)$ . Then,  $HF^+(Y, \mathfrak{s}) = \mathbb{Z}$  if  $\mathfrak{s} = \mathfrak{s}_0$  where  $\mathfrak{s}_0$  is the  $\text{spin}^c$  structure corresponding to vertical tangent bundle and  $HF^+(Y, \mathfrak{s}) = 0$  otherwise.*

*Proof.* Since the diagram is weakly admissible, we can let  $t = 1$  and the result follows from the previous theorem.  $\square$

In general, let  $\partial_{\text{right}}^+$  be the contribution to the Heegaard Floer differential from the holomorphic disks whose domain lies in  $\bar{F}$  (i.e. the disks which lie on the right half of our almost admissible Heegaard diagrams), also let  $CF_{\text{right}}^+ = C_{\text{right}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$ , the chain complex associated with the right side of the diagram for the purpose of constructing  $HF^+$  theory.

**Corollary 16**  *$(CF_{\text{right}}^+, \partial_{\text{right}}^+)$  is a chain complex with rank 1 homology generated by  $\mathbf{x}_1$ .*

*Proof.* This is only a reformulation of the above results.  $\square$

### 3 The isomorphism

In this section, we prove the main theorem of this paper. Namely, we prove that the perturbed Heegaard Floer homology group  $HF^+(Y, f, \gamma_w)$  is isomorphic to the Floer homology of the chain complex  $(C_{\text{left}}, \partial_{\text{left}}; \Lambda)$ . Before stating our theorem let us digress to give a rigorous definition of the latter chain complex.

#### 3.1 A variant of Heegaard Floer homology for broken fibrations over the circle

Let  $Y$  be a 3-manifold with  $b^1 > 0$ , and let  $f : Y \rightarrow S^1$  be a broken fibration with connected fibres, and satisfying the conditions at the beginning of Section 2.1. As before consider the highest genus fibre  $\Sigma_g$  and let  $\alpha_1, \dots, \alpha_{g-k}$  and  $\beta_1, \dots, \beta_{g-k}$  be tuples of  $g - k$  disjoint linearly independent simple closed curves on  $\Sigma_g$  obtained from the attaching circles corresponding to the critical values of  $f$ , and let  $w$  be a base point that is in the complement of  $\alpha$  and  $\beta$  curves. As in Lemma 6, we can arrange by winding if necessary that there are no periodic domains. We define the Floer homology of such a configuration in a manner similar to the usual Heegaard Floer theory by defining the chain complex to be the  $\Lambda$ -module freely generated by intersection points of  $\mathbb{T}_{\alpha}^{g-k} = \alpha_1 \times \dots \times \alpha_{g-k}$  and  $\mathbb{T}_{\beta}^{g-k} = \beta_1 \times \dots \times \beta_{g-k}$  in  $\text{Sym}^{g-k}(\Sigma_g)$ , equipped with a differential given as follows:

$$\partial \mathbf{x} = \sum_{\varphi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\varphi)=1} \# \widehat{\mathcal{M}}(\varphi) r^{n_w(\varphi)} \mathbf{y}$$

For reasons that will be clarified in Section 4, we will denote the homology group that we expect to get from this construction  $QFH'(Y, f; \Lambda)$ . This stands for *quilted Floer homology* of the broken fibration  $(Y, f)$  with coefficients in  $\Lambda$ . There are at least two obvious issues that we need to address in order to make sure that  $QFH'(Y, f; \Lambda)$  is well-defined. The first issue is the compactness of the moduli space  $\mathcal{M}(\varphi)$ . The second issue is proving that  $\partial^2 = 0$ . The setup here is more delicate than the usual setup of Heegaard Floer homology due to the fact that  $\text{Sym}^{g-k}(\Sigma_g)$  is not a (positively) monotone symplectic manifold when  $k > 0$  (it has  $\langle c_1, [\Sigma_g] \rangle = 2 - 2k$ ). Therefore, one expects the existence of configurations with negative Chern number bubbles. However, we will adopt the cylindrical setting of Lipshitz ([12]), whereby one considers pseudo-holomorphic curves in  $\Sigma_g \times [0, 1] \times \mathbb{R}$  instead of disks in  $\text{Sym}^{g-k}(\Sigma_g)$ , and choose our almost complex structures from a sufficiently general class. Namely, one chooses a translation-invariant almost-complex structure  $J$  on  $\Sigma_g \times [0, 1] \times \mathbb{R}$  such that  $J$  preserves a 2-plane distribution  $\xi$  on  $\Sigma_g \times [0, 1]$  which is tangent to  $\Sigma_g$  near  $(\alpha \cup \beta) \times [0, 1]$  and near  $\Sigma_g \times \partial[0, 1]$  (see [12], axiom J5'). Now we can show that transversality can be achieved for holomorphic curves in the homology class of the fibre of the projection  $\Sigma_g \times [0, 1] \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}$ . However the expected dimension of these curves is negative, hence bubbling at interior points can be ruled out a priori (see [12] Lemma 8.2). Furthermore, since we assumed that all the fibres are connected, the  $(g - k)$ -tuples of curves are linearly independent in homology; this implies that any boundary bubble lifts to a spherical class in  $\pi_2(\text{Sym}^{g-k}(\Sigma_g))$ . By choosing almost complex structures in a specific way as in [12] Lemma 8.2, we can also avoid boundary bubbles. Thus the compactness of  $\mathcal{M}(\varphi)$  is ensured.

The drawback of this approach is that it does not correspond in a straightforward way to the original setting in  $(\text{Sym}^{g-k}\Sigma_g, \mathbb{T}_\alpha^{g-k}, \mathbb{T}_\beta^{g-k})$  since such general almost complex structures prevent the fibres of the projection to  $[0, 1] \times \mathbb{R}$  from being complex. In this case, in order to be able work in  $\text{Sym}^{g-k}(\Sigma_g)$  one needs to establish a proper combinatorial rule for handling bubbled configurations (for example by applying the general machinery of virtual fundamental cycles [14]). It is reasonable to expect that one would then get the same differential as above, but the argument would be technically very involved. However, there is an exception to this, namely when we are in the strongly negative case, that is when  $g < 2k$ . We show in Section 4 that in this case we can indeed use integrable complex structures of the form  $\text{Sym}^{g-k}(j_s)$  for a path  $j_s$  of complex structures on  $\Sigma$  and still avoid bubbling by making use of the Abel-Jacobi map.

The proof of  $\partial^2 = 0$  for  $QFH'(Y, f; \Lambda)$  will be part of the proof of the isomorphism that we will construct between  $QFH'(Y, f; \Lambda)$  and  $HF^+(Y, f, \gamma_w)$ . Namely, this follows from an identification between the Maslov index 1 moduli spaces in both theories. Furthermore, we will also see in this section that  $QFH'(Y, f; \Lambda)$  is an invariant of  $(Y, [f])$ , that is it only depends on the homotopy class of  $f$  and, when defined over  $\mathbb{Z}$ , it will be an invariant of  $Y$ .

As usual in Floer homology theories, the groups  $QFH'(Y, f; \Lambda)$  are graded by equivalence classes of  $\text{spin}^c$  structures. Given an intersection point in  $\mathbf{x} \in \mathbb{T}_\alpha^{g-k} \cap \mathbb{T}_\beta^{g-k}$  one gets a  $\text{spin}^c$  structure

$\mathfrak{s}(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$ , as in Heegaard Floer theory, except we do not need to consider any additional base point since the intersection point  $\mathbf{x}$  gives a matching of index 1 and 2 critical points of  $f$ , which in turn determines a  $\text{spin}^c$  structure by taking the gradient vector field of  $f$  outside of tubular neighborhoods of these matching flow lines and extending it in a non-vanishing way to the tubular neighborhoods. We remark that in our setup of Heegaard diagram for  $(Y, f)$ , we have the equality  $\mathfrak{s}(\mathbf{x}_{\text{left}}) = \mathfrak{s}_z(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1)$  (where  $\mathbf{x}_1$  is as in Lemma 13).

**Remark:** Note that if we restrict to the case where we only count  $n_w = 0$  curves and forget the data encoded in the  $\alpha$  and  $\beta$  that do not come from the critical values of  $f$ , we obtain Juhász's sutured Floer homology groups associated with the diagram  $(F, \alpha_1, \dots, \alpha_{g-k}, \beta_1, \dots, \beta_{g-k})$  (see [6]). We will return to this below.

### 3.2 Isomorphism between $QFH'(Y, f; \Lambda)$ and $HF^+(Y, f, \gamma_w)$

We now proceed to prove an isomorphism between  $QFH'(Y, f; \Lambda)$  and  $HF^+(Y, f, \gamma_w)$ . As a first step, we make use of the calculations of the previous section. Let  $CF_{\text{left}}^+ = C_{\text{left}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$  and  $CF_{\text{right}}^+ = C_{\text{right}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$ , using the splitting of generators of  $HF^+(Y, f, \gamma_w)$  as discussed in Section 2.2, so that we have  $CF^+(Y, f, \gamma_w) = CF_{\text{left}}^+ \otimes CF_{\text{right}}^+$ . We denote by  $\partial_F$  and  $\partial_{\bar{F}} = \mathbb{1} \otimes \partial_{\text{right}}$  the contributions to the Heegaard Floer differential from holomorphic curves whose domains lie in  $F$  and  $\bar{F}$  respectively. Furthermore, we denote by  $\partial_{\text{left}} \otimes \mathbb{1}$ , the contribution of those holomorphic curves whose domain lies in  $F$  and which act by identity on  $C_{\text{right}}$  with respect to the splitting  $C_{\text{left}} \otimes C_{\text{right}}$ . (since the boundary of  $F$  includes points of intersections occurring in  $C_{\text{right}}$ , this is a priori more restrictive than  $\partial_F$ ). Lemma 13 implies that  $\partial_{\text{left}} \otimes \mathbb{1}$  is a differential on  $C_{\text{left}} \otimes \mathbf{x}_1$ . The next proposition says that the homology of this differential is isomorphic to  $HF^+(Y, f, \gamma_w)$ .

**Proposition 17**  $HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq H(C_{\text{left}} \otimes \mathbf{x}_1, \partial_{\text{left}} \otimes \mathbb{1}, \gamma_w, \mathfrak{s})$  for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ .

*Proof.* Both homology groups are filtered by  $n_w$ . Therefore, there are spectral sequences converging to both sides induced by the  $n_w$  filtration. Furthermore, we claim that there is a chain map:

$$F : C_{\text{left}} \otimes \mathbf{x}_1 \rightarrow CF_{\text{left}}^+ \otimes CF_{\text{right}}^+$$

given by

$$F(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1) = [\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, 0]$$

which induces an isomorphism of  $E^1$ -pages of the spectral sequences associated with both chain complexes. The fact that  $F$  is a chain map, is a consequence of Lemma 13. More precisely, Lemma 13 gives that if a holomorphic map contributing to the differential originates at  $[\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, 0]$  then it has to converge to a generator of the form  $[\mathbf{y}_{\text{left}} \otimes \mathbf{x}_1, 0]$ , and the domain of the map has to lie on the left half of the Heegaard diagram; these are exactly the contributions to the differential captured by  $\partial_{\text{left}} \otimes \mathbb{1}$ .

Furthermore, showing that  $F$  induces an isomorphism on the  $E^1$ -pages of the spectral sequences on both sides amounts to checking that

$$F' : (C_{\text{left}} \otimes \mathbf{x}_1, \partial_{\text{left}}^0 \otimes \mathbb{1}) \rightarrow (CF_{\text{left}}^+ \otimes CF_{\text{right}}^+, \partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})$$

is an isomorphism in homology, where  $\partial_{\text{left}}^0 \otimes \mathbb{1}$  denotes those holomorphic maps contributing to the differential  $\partial_F$  with  $n_w = 0$  (Here we have used Lemma 10 to identify  $n_w = 0$  part of  $\partial^+$  with  $\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$ ). The injectivity of  $F'$  in homology follows from the fact that, by Corollary 16 (see also the proof of Theorem 14),  $\mathbf{x}_1$  does not lie in the image of  $\partial_{\text{right}}$ . Thus, we only need to check that  $F'$  is surjective in homology. Suppose that  $\mathbf{a}_1 \mathbf{x}_1 + \dots + \mathbf{a}_{4k} \mathbf{x}_{4k} + \mathbf{b}_1 \mathbf{y}_1 + \dots + \mathbf{b}_{4k} \mathbf{y}_{4k} \in CF_{\text{left}}^+ \otimes CF_{\text{right}}^+$  is in the kernel of  $\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$ , where we have chosen the notation so that  $\mathbf{a}_i$  and  $\mathbf{b}_i$  are elements in  $CF_{\text{left}}^+ = C_{\text{left}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$ , and  $\mathbf{x}_i$  and  $\mathbf{y}_i$  are the generators of  $C_{\text{right}}$  as in Theorem 14. Now, because this element is in the kernel of  $\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$ , we have

$$\begin{aligned} \partial_{\text{left}}^0 \mathbf{a}_1 &= 0 & \text{and} & & U \mathbf{a}_1 + \partial_{\text{left}}^0 \mathbf{b}_1 &= 0 \\ \partial_{\text{left}}^0 \mathbf{a}_i &= 0 & \text{and} & & \mathbf{a}_i + \partial_{\text{left}}^0 \mathbf{b}_i &= 0 \quad \text{for } i \neq 1 \end{aligned}$$

where  $U : CF_{\text{left}}^+ \rightarrow CF_{\text{left}}^+$  is the usual map in Heegaard Floer theory which maps  $[\mathbf{a}, i] \rightarrow [\mathbf{a}, i-1]$ . It appears in the above equation because the disk  $D_1$  connecting  $\mathbf{x}_1$  to  $\mathbf{y}_1$  intersects the base point  $z$  with multiplicity 1. (Here we also chose an orientation system so that  $\partial_{\text{right}} \mathbf{x}_i = \mathbf{y}_i$ , one can also do the same calculation if  $\partial_{\text{right}} \mathbf{x}_i = -\mathbf{y}_i$ .)

Now, observe that the above equations give

$$(\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})(\mathbf{b}_i \mathbf{x}_i) = -\mathbf{a}_i \mathbf{x}_i + \mathbf{b}_i \mathbf{y}_i \quad \text{for } i \neq 1$$

This gives us that  $2\mathbf{b}_i \mathbf{y}_i$  is in the kernel, which in turn, implies that  $\partial_{\text{left}}^0 \mathbf{b}_i = 0$  (This holds unless we work over a field of characteristic 2, see below for that case). Thus,  $\mathbf{a}_i = 0$  and  $\mathbf{b}_i \mathbf{y}_i$  is in the image of  $\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$  (In characteristic 2, we directly conclude that  $\mathbf{a}_i \mathbf{x}_i + \mathbf{b}_i \mathbf{y}_i$  is in the image). Therefore, in either case we can ignore all the terms other than  $\mathbf{a}_1 \mathbf{x}_1 + \mathbf{b}_1 \mathbf{y}_1$ . Furthermore, note that

$$(\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})(U^{-1} \mathbf{b}_1 \mathbf{x}_1) = U^{-1} \partial_{\text{left}}^0 \mathbf{b}_1 \mathbf{x}_1 + \mathbf{b}_1 \mathbf{y}_1 = -\mathbf{a}_1 \mathbf{x}_1 + \mathbf{b}_1 \mathbf{y}_1$$

Thus, we conclude that  $2\mathbf{b}_1 \mathbf{y}_1$  is in the kernel, which implies that  $\partial_{\text{left}}^0 \mathbf{b}_1 = 0$  hence,  $U \mathbf{a}_1 = 0$  and  $(\partial_{\text{left}}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})(U^{-1} \mathbf{b}_1 \mathbf{x}_1) = \mathbf{b}_1 \mathbf{y}_1$  hence we can ignore the term  $\mathbf{b}_1 \mathbf{y}_1$  and the fact that  $U \mathbf{a}_1 = 0$  implies that  $\mathbf{a}_1 \mathbf{x}_1$  is in the image of  $F$  as desired.

This concludes the proof of Proposition 17 since a chain map that induces an isomorphism of  $E^1$ -pages induces an isomorphism at all pages of the spectral sequences (see e.g. Theorem 3.5 of [16]), in particular the  $E^\infty$ -pages are the groups that we have written in the statement of Proposition 17.  $\square$

An immediate corollary that follows from the proof of Proposition 17 is that the  $U$ -action on  $HF^+(Y, f, \gamma_w, \mathfrak{s})$  is trivial for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ . In fact, we have a splitting of the long exact sequence induced by the  $U$ -action, which implies the following relation with the hat-version of Heegaard Floer homology where the differential counts the holomorphic curves with  $n_z = 0$  (see [18]).

**Corollary 18** For  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ ,  $\widehat{HF}(Y, f, \gamma_w, \mathfrak{s}) \simeq HF^+(Y, f, \gamma_w, \mathfrak{s}) \oplus HF^+(Y, f, \gamma_w, \mathfrak{s})[1]$   $\square$

Note that in the case that  $g(\Sigma_{\min}) = k > 1$ , there is no perturbation required thus the above equality holds for the homology groups with integer coefficients. In particular, this implies that  $HF^+(Y, \mathfrak{s})$  is algorithmically computable for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$  since in [27] an algorithm for computing  $\widehat{HF}(Y, \mathfrak{s})$  was given.

Finally, we are ready to state and prove our main result. Over the course of the proof of the following theorem, we will see why the variant of Heegaard Floer homology that we denoted by  $QFH'(Y, f, \mathfrak{s}; \Lambda)$  is well-defined. More precisely, we will see that the differential that we defined for  $QFH'(Y, f, \mathfrak{s}; \Lambda)$  squares to zero.

**Theorem 19**  $HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f, \mathfrak{s}; \Lambda)$  for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ .

*Proof.* Because of Proposition 17, it suffices to prove that

$$H(C_{\text{left}} \otimes \mathbf{x}_1, \partial_{\text{left}} \otimes \mathbb{1}, \gamma_w, \mathfrak{s}) \simeq H(C_{\text{left}}, \partial, \mathfrak{s})$$

where the latter group is what we previously called  $QFH(Y, f, s)$ . Clearly, we have a one-to-one correspondence between the generators. Next, we will show that there is an isomorphism of chain complexes. In fact, we will show that the signed counts of Maslov index 1 holomorphic curves in  $\pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$  which contribute to  $\partial_{\text{left}} \otimes \mathbb{1}$  and Maslov index 1 holomorphic curves in  $\pi_2(\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}})$  that contribute to the differential  $\partial$  are equal. First observe that for curves which stay away from the necks at  $\alpha_0$  and  $\beta_0$ , which are precisely those with  $n_w = 0$ , this one to one correspondence is clear. (These are the curves that contribute to the differential  $\partial_{\text{left}}^0 \otimes \mathbb{1}$  in Proposition 17).

Next, we discuss the curves which have  $n_w \neq 0$ . The correspondence in this case will be obtained by stretching the necks along  $a$  and  $b$ , which are respectively parallel pushoffs of  $\alpha_0$  and  $\beta_0$  to the left of the Heegaard diagram (into the region  $F$ ).

Let us first describe the holomorphic curves that contribute to  $\partial_{\text{left}} \otimes \mathbb{1}$  with  $n_w \neq 0$  more precisely. Remember that by definition  $\partial_{\text{left}} \otimes \mathbb{1}$  counts those holomorphic curves whose domain lies in  $F$ , hence they have  $n_z = 0$ . Now, recall that Lemma 10 says that the projection to the Heegaard surface is an unbranched cover around the necks  $a$  and  $b$ . Let  $A \in \pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$  be a Maslov index 1 homology class which is contributing to  $\partial_{\text{left}} \otimes \mathbb{1}$ . By degenerating the almost complex structure around  $a$  and  $b$  on  $\Sigma$ , we get two homology classes  $A_{\text{left}} \in \pi_2(\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}})$  and

$A_{\text{right}} \in \pi_2(\mathbf{x}_1, \mathbf{x}_1)$ . The domain of  $A_{\text{left}}$  lies on  $\Sigma_{\text{max}}$  and it determines a homology class for the type of holomorphic curves contributing to the differential  $\partial$ . The domain of  $A_{\text{right}}$  has two components  $A_{\text{right}}^a$  and  $A_{\text{right}}^b$ , both supported in disks which are the domains between  $\alpha_0$  and  $a$ , with  $a$  collapsed to a point, and between  $\beta_0$  and  $b$  with  $b$  collapsed to a point. We claim that the Maslov index of  $A_{\text{left}}$  is equal to 1, and the Maslov indices of each of the components in  $A_{\text{right}}$  are equal to  $2n_w$ . Since the degeneration is along Reeb orbits, we have the formula

$$\text{ind}(A) = \text{ind}(A_{\text{left}}) + \text{ind}(A_{\text{right}}^a) + \text{ind}(A_{\text{right}}^b) - 2(N_a + N_b)$$

where  $N_a$  and  $N_b$  are the numbers of connected components of the unramified covering in the necks at  $a$  and  $b$  (clearly  $N_a, N_b \in [1, n_w]$ ). Therefore, it suffices to see that  $\text{ind}(A_{\text{right}}^a) = \text{ind}(A_{\text{right}}^b) = 2n_w$ . This follows from the usual formula  $\text{ind}(A_{\text{right}}^a) = \langle c_1(\mathfrak{s}), A_{\text{right}}^a \rangle = e(A_{\text{right}}^a) + 2n_{\mathbf{x}}(A_{\text{right}}^a) = 2n_w$  (since the homology class  $A_{\text{right}}^a$  is  $n_w$  times the disk with boundary on  $\alpha_0$ ,  $e(A_{\text{right}}^a) = n_w$  and  $n_{\mathbf{x}}(A_{\text{right}}^a) = n_w/2$ ); similarly for  $A_{\text{right}}^b$ . We deduce that  $\text{ind}(A_{\text{left}}) = 1 + 2(N_a + N_b) - 4n_w$ , which implies that  $\text{ind}(A_{\text{left}}) = 1$  and the coverings in the cylindrical necks near  $a$  and  $b$  are both trivial (in other terms, after neck-stretching we have  $n_w$  distinct cylinders passing through each neck).

Furthermore, we have the evaluation maps :

$$\begin{aligned} \text{ev}_{\text{left}}^a &: \mathcal{M}(A_{\text{left}}) \rightarrow \text{Sym}^{n_w}(\mathbb{D}) \\ \text{ev}_{\text{right}}^a &: \mathcal{M}(A_{\text{right}}^a) \rightarrow \text{Sym}^{n_w}(\mathbb{D}) \\ \text{ev}_{\text{left}}^b &: \mathcal{M}(A_{\text{left}}) \rightarrow \text{Sym}^{n_w}(\mathbb{D}) \\ \text{ev}_{\text{right}}^b &: \mathcal{M}(A_{\text{right}}^b) \rightarrow \text{Sym}^{n_w}(\mathbb{D}) \end{aligned}$$

given by taking the preimages of the degeneration points of  $a$  and  $b$  and projecting to  $\mathbb{D} = [0, 1] \times \mathbb{R}$ . We claim that the moduli space  $\mathcal{M}(A)$  can be identified with the fibre product of moduli spaces  $\mathcal{M}(A_{\text{left}}) \times_B \mathcal{M}(A_{\text{right}})$ , where  $B = \text{Sym}^{n_w}(\mathbb{D}) \times \text{Sym}^{n_w}(\mathbb{D})$  and the fibre product is taken with respect to the above evaluation maps. This is a consequence of a gluing theorem (see [19] Theorem 5.1 for the proof in a very closely related situation and [3] for a discussion of gluing in a general context).

Finally, we will prove that  $(\text{ev}_{\text{right}}^a, \text{ev}_{\text{right}}^b) : \mathcal{M}(A_{\text{right}}^a) \times \mathcal{M}(A_{\text{right}}^b) \rightarrow \text{Sym}^{n_w}(\mathbb{D}) \times \text{Sym}^{n_w}(\mathbb{D})$  has degree 1. This implies that, for the purpose of counting pseudoholomorphic curves, the fibre product of moduli spaces  $\mathcal{M}(A_{\text{left}}) \times_B \mathcal{M}(A_{\text{right}})$  can be identified with  $\mathcal{M}(A_{\text{left}})$ . Therefore, we can identify the moduli spaces  $\mathcal{M}(A)$  and  $\mathcal{M}(A_{\text{left}})$ , as required.

To see that the evaluation maps have degree 1, we argue as follows: First, we represent the domain of the strips in  $\mathcal{M}(A_{\text{right}}^a)$  by the upper half of the unit disk so that the upper half circle maps to  $\alpha_0$  and the interval  $[-1, 1]$  maps to the  $\beta$  curve. Also, represent the target disk by the unit disk, so that  $\alpha_0$  corresponds to the unit circle and the  $\beta$  arc is represented by the real positive axis, furthermore the degeneration point of  $a$  as used to define the map  $\text{ev}_{\text{right}}^a$  is mapped to the origin in

this representation. Thus, the moduli space  $\mathcal{M}(A_{\text{right}}^a)$  consists of holomorphic maps from the upper half disk to the unit disk and  $ev_{\text{right}}^a$  records the positions of the  $n_w$  zeroes of these maps. Now, any holomorphic map from the upper half disk to the unit disk can be reflected ( $u(1/\bar{z}) := 1/\overline{u(z)}$ ) to get a holomorphic map from the upper half-plane to  $\mathbb{P}^1$ , mapping the real axis to the real positive axis. This can then be further reflected about the real axis to get holomorphic maps from  $\mathbb{P}^1$  to  $\mathbb{P}^1$  which are hence rational fractions of degree  $2n_w$ , with real coefficients (forced by the invariance under conjugation) and with equivariance under  $z \rightarrow 1/\bar{z}$ . Now, such holomorphic maps are classified by their zeroes (the poles are the reflections of the zeroes). In our case, there are  $2n_w$  zeroes and none of these are real, so they are  $n_w$  pairs of complex conjugate points. Finally, we note that  $ev_{\text{right}}^a$  maps any such holomorphic map to the positions of its  $n_w$  zeroes which lie inside the upper half-disk. Therefore,  $ev_{\text{right}}^a : \mathcal{M}(A_{\text{right}}^a) \rightarrow \text{Sym}^{n_w}(\mathbb{D})$  is in fact a diffeomorphism. In particular, it has degree 1.  $\square$

Note that when the minimal genus fibre has genus greater than 1, there is no perturbation required since the diagrams that we consider are weakly admissible in that case. Hence, we get the above isomorphism for the homology groups with integer coefficients.

**Corollary 20** *Suppose that  $g(\Sigma_{\min}) = k > 1$ , then for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$  we have*

$$QFH^i(Y, f, \mathfrak{s}; \mathbb{Z}) \simeq HF^+(Y, \mathfrak{s}).$$

*Proof.* This follows from the above result by letting  $t = 1$ .  $\square$

In some cases, the quilted Floer homology groups can be calculated easily, the following special case is an example of this. Given two simple closed curves  $\alpha$  and  $\beta$  on a surface of genus greater than 1, let  $\iota(\alpha, \beta)$  denote the geometric intersection number of  $\alpha$  and  $\beta$ , i.e. the number of transverse intersections of their geodesic representatives for a hyperbolic metric.

**Corollary 21** *Suppose that  $f$  has only two critical points, and let  $\alpha, \beta \subset \Sigma_{\max}$  be the vanishing cycles for these critical points. Then  $\bigoplus_{\mathfrak{s} \in \mathcal{S}(\Sigma_{\min})} HF^+(Y, f, \gamma_w, \mathfrak{s})$  is free of rank  $\iota(\alpha, \beta)$ .*

*Proof.* When  $f$  has only two critical points,  $QFH^i(Y, f)$  reduces to the Lagrangian Floer homology of the simple closed curves  $\alpha$  and  $\beta$  on the surface  $\Sigma_{\max}$ . This is easily calculated by representing the free homotopy classes of simple closed curves  $\alpha$  and  $\beta$  by geodesics, which ensures that there are no non-constant holomorphic discs contributing to the differential. In fact, any holomorphic disk would lift to a holomorphic disk in the universal cover  $\mathbf{H}^2$ , which would contradict the fact that there is a unique geodesic between any two points in  $\mathbf{H}^2$ . Therefore, the quilted Floer homology is freely generated by the number of intersection points of geodesic representatives of  $\alpha$  and  $\beta$ .  $\square$

We remark that if  $\iota(\alpha, \beta) = 1$ , then the critical values can be cancelled. Thus for non-fibred manifolds which admit a broken fibration with only 2 critical points the rank of quilted Floer homology is greater than 1.

### 3.3 An application to sutured Floer homology

The following definition of a sutured 3–manifold can be easily seen to be equivalent to the standard definition (see Juhász [6]). A connected balanced sutured manifold is a compact oriented 3–manifold with boundary  $Y$  such that  $Y$  can be equipped with a broken fibration  $f : Y \rightarrow [0, 1]$  whose fibers are surfaces with non-empty boundary and  $f^{-1}(0)$  and  $f^{-1}(1)$  are homeomorphic surfaces such that each connected component has exactly one boundary component (balanced condition). We can always arrange that  $f^{-1}(1/2) = \Sigma_{\max}$  is the highest genus fibre which is connected and as one travels from  $1/2$  to  $0$  one attaches two handles along  $\beta_1, \dots, \beta_{g-k}$  and as one travels from  $1/2$  to  $1$  one attaches two handles along  $\alpha_1, \dots, \alpha_{g-k}$  which are realized as vanishing cycles of  $f$  on  $\Sigma_{\max}$ . The balanced condition translates to the condition that the set of  $\alpha$  curves and respectively the set of  $\beta$  curves are linearly independent in  $H_1(\Sigma_{\max})$ . The sutures  $s(\gamma)$  of  $Y$  correspond to the boundary components of  $\partial\Sigma_{\max}$  and the annular neighborhoods  $A(\gamma)$  of  $Y$  are obtained from  $s(\gamma)$  by flowing using the gradient flow of  $f$  along  $[0, 1]$  with respect to a metric such that the gradient vector field of  $f$  preserves the boundary of  $Y$ .

In [6], Juhász constructs a variant of Heegaard Floer homology for sutured 3–manifolds. This is simply, the Lagrangian Floer homology group  $HF(\text{Sym}^{g-k}(\Sigma_{\max}), \alpha_1 \times \dots \times \alpha_{g-k}, \beta_1 \times \dots \times \beta_{g-k})$  where the projections of the holomorphic curves contributing to the differential on  $\Sigma_{\max}$  are required to stay away from the boundary of  $\Sigma_{\max}$ .

In [7], Kronheimer and Mrowka construct an invariant of sutured manifolds using monopole (resp. instanton) Floer homology, by constructing a closed 3–manifold  $Y_n$  and setting the sutured Floer homology of  $Y$  by defining it to be the monopole (resp. instanton) Floer homology of  $Y_n$ . The construction of  $(Y_n, f_n)$  is by first gluing  $T \times [0, 1]$  where  $T$  is an oriented connected genus  $n \geq 1$  surface with non-empty boundary, so that  $\partial T \times [0, 1]$  is glued to the union of annuli  $A(\gamma)$ , and then identifying the fibres over  $0$  and  $1$  by choosing a homeomorphism between them. Note that the balanced condition implies that  $f_n$  has connected fibres. In the monopole (resp. instanton) setting, Kronheimer and Mrowka define the sutured monopole (resp. instanton) Floer homology of  $Y$  to be  $\bigoplus_{s \in \mathcal{S}(Y|\Sigma_{\min})} HM(Y_n, \mathfrak{s})$  and prove that this is an invariant of the sutured manifold  $Y$  (in particular, it is also independent of the genus  $n$  of  $T$  and the homeomorphism chosen in identifying fibres over  $0$  and  $1$ ). It was raised in [7] as a question, whether one can recover Juhász’s definition of sutured Floer homology from the construction given above applied in the setting of Heegaard Floer homology. In the next theorem, we give an affirmative answer to this.

**Theorem 22** For  $n \geq 1$ ,

$$SFH(Y, f) \simeq \bigoplus_{s \in \mathcal{S}(Y_n, \Sigma_{\min})} HF^+(Y_n, \mathfrak{s})$$

Note that this theorem in particular implies that the group on the right hand side is independent of  $n$  and the chosen surface homeomorphism in the construction of  $Y_n$ . As usual, in the case that the

lowest genus fibre of  $f_1$  has genus 1, one needs to use coefficients in  $\Lambda$ .

*Proof.* Theorem 19 applied to  $(Y_n, f_n)$  yields that  $\bigoplus_{\mathfrak{s} \in \mathcal{S}(Y_n, \Sigma_{\min})} HF^+(Y_n, \mathfrak{s}) = QFH'(Y_n, f_n)$ . Therefore, the proof will follow once we establish that  $SFH(Y, f) \simeq QFH'(Y_n, f_n)$ . This in turn relies on a simple observation about the Heegaard diagrams used in the definition of these groups, namely let us denote the maximal genus fibre of  $f$  by  $\Sigma$ , and the maximal genus fibre of  $f_n$  by  $\Sigma \cup T$ . Now, if an admissible sutured Heegaard diagram of  $(Y, f)$  is given by  $(\Sigma, \alpha_1, \dots, \alpha_{g-k}, \beta_1, \dots, \beta_{g-k})$ , then the Heegaard diagram for calculating  $QFH'(Y_n, f_n)$  is given by  $(\Sigma \cup T, \alpha_1, \dots, \alpha_{g-k}, \beta_1, \dots, \beta_{g-k})$ . Note that there is no  $\alpha$  or  $\beta$  curve entering  $T$ . Thus, the proof will be complete once we prove that holomorphic curves contributing to the differential of  $QFH'(Y_n, f_n)$  do not enter to the region including  $T$ . Note that because of the admissibility condition of the sutured Heegaard diagram of  $(Y, f)$  we can use an almost complex structure which is vertical in a neighborhood of  $\Sigma \times [0, 1] \times \mathbb{R}$  so that the holomorphic curves contributing to the differential of sutured Floer homology appear as holomorphic curves contributing to the differential of  $QFH'(Y_n, f_n)$ . On the other hand, we use a non-vertical almost complex structures as in Section 3.1, along  $T \times [0, 1] \times \mathbb{R}$  away from the boundary of  $T$ . Now, let  $u : (S, \partial S) \rightarrow (\Sigma \cup T) \times [0, 1] \times \mathbb{R}$  be a holomorphic map contributing to the differential of  $QFH'(Y_n, f_n)$ . We would like to show that the image of the projection of  $u$  to the Heegaard surface does not hit  $T$ . This follows from a degeneration argument. Namely, suppose that the image of the projection of  $u$  does hit  $T$ , then we can degenerate along Reeb orbits corresponding to the attaching region of  $T$  to  $\Sigma$ , this would on one side give a holomorphic map  $\tilde{u} : \tilde{S} \rightarrow \tilde{T} \times [0, 1] \times \mathbb{R}$  where  $\tilde{T}$  is the closed surface obtained by shrinking each boundary component of  $T$  to a point and  $\tilde{S}$  is a part of the domain of the degenerated map. The index formula (see for example [12]) gives that the expected dimension of the moduli space of such maps is a positive multiple of  $\chi(\tilde{T})$ , which is less than or equal to zero since  $T$  has genus at least 1. Furthermore, our choice of almost complex structures ensure transversality for such holomorphic maps, which yields the desired contradiction (note that in the case that  $T$  has genus 1, we still obtain a contradiction since we get a negative dimension for the transversely cut moduli space after quotienting by the  $\mathbb{R}$  action).  $\square$

#### 4 Isomorphism between $QFH(Y, f; \Lambda)$ and $QFH'(Y, f; \Lambda)$

In this section, we relate  $QFH'(Y, f)$  defined as a variant of Heegaard Floer homology as in Section 3.1 with the original definition in terms of holomorphic quilts given in the introduction, which we called  $QFH(Y, f)$ . The arguments given here are complete except in the statement of the main theorem (see Theorem 25) we make an assumption about the non-existence of figure-eight bubbles. This is a new kind of bubbling that arises in the study of holomorphic quilts when the width of a strip is shrunk to zero (see [28] for more background on this). Under suitable strong negativity assumptions, when  $g < 2k$ , we argue that figure-eight bubbles do not arise by assuming a removal

of singularities result. The assumption  $g < 2k$  is required in order to avoid disk and sphere bubbles so that the group  $QFH(Y, f; \Lambda)$  is well defined. The new input here is that this condition is also sufficient to discard figure-eight bubbles. However, a removal of singularities theorem for figure-eight bubbles is missing at the time of this writing. It appears likely that this is actually not an issue in the setting we consider; we hope to return to this at a later time. Until then when discussing our results we make our arguments under the assumption that the figure-eight bubbles do not arise.

Finally, we remark that all the theorems are stated for Floer homology groups over the universal Novikov ring  $\Lambda$ , but as before, in the case where the lowest genus fibre has genus greater than 1, one can use integer coefficients.

#### 4.1 Heegaard tori as composition of Lagrangian correspondences

Recall that given a broken fibration  $f : Y \rightarrow S^1$  with connected fibres, the quilted Floer homology of  $Y$  is defined as the Floer homology of the Lagrangian correspondences  $L_{\alpha_1}, \dots, L_{\alpha_{g-k}}$  and  $L_{\beta_1}, \dots, L_{\beta_{g-k}}$  associated with the critical values of  $f$ , where as before we let the vanishing cycles for the critical values along the northern semi-circle be  $\alpha_1, \dots, \alpha_{g-k}$  and those along the southern semi-circle be  $\beta_1, \dots, \beta_{g-k}$ . Let us call the Floer homology of these Lagrangian correspondences  $HF(\mathbb{L}_\alpha, \mathbb{L}_\beta)$ . Note that the construction of these Lagrangians involves a choice of almost complex structure  $j$  on the fibres of  $f$ .

Recall that given two Lagrangian correspondences,  $L_1 \subset X \times Y$  and  $L_2 \subset Y \times Z$  such that  $L_1 \times L_2$  is transverse to the diagonal in  $Y$ , the composition  $L_1 \circ L_2$  is a Lagrangian correspondence in  $X \times Z$  given by the union of tuples  $(x, z)$  such that there exists a  $y \in Y$  with the property that  $(x, y) \in L_1$  and  $(y, z) \in L_2$ .

Now, for the class of almost complex structures  $j$  that are sufficiently stretched along the vanishing cycles of  $f$  near its critical points, we have the following important technical lemma about these correspondences which was conjectured by Perutz in [23]:

**Lemma 23** *For  $g > k$ ,  $L_{\alpha_1} \circ \dots \circ L_{\alpha_{g-k}}$  and  $L_{\beta_1} \circ \dots \circ L_{\beta_{g-k}}$  are respectively Hamiltonian isotopic to  $\alpha_1 \times \dots \times \alpha_{g-k}$  and  $\beta_1 \times \dots \times \beta_{g-k}$  in  $\text{Sym}^{g-k}(\Sigma)$  equipped with a Kähler form  $\omega$  which lies in the cohomology class  $\eta + \lambda\theta$  with  $\lambda > 0$ .  $\square$*

Here the classes  $\eta$  and  $\theta$  are cohomology classes which generate the subspace of cohomology classes which are invariant under the action of the mapping class group.  $\eta$  is the Poincaré dual of the divisor  $\{pt\} \times \text{Sym}^{g-k-1}(\Sigma)$  and  $\theta$  corresponds to the intersection form on  $H^2(\Sigma, \mathbb{R})$  via the Abel-Jacobi map, more precisely it is the pullback by the Abel-Jacobi map of the theta divisor on the Jacobian.

This proof of this lemma can be found in [11] and [10]. The proof is obtained by carrying out the construction of Lagrangian correspondences as a family of degenerations. As the required technical set-up is developed in [11], for the sake of brevity we choose to omit it from here.

Recall that when defining  $QFH'(Y, f, \Lambda)$  as a variant of Heegaard Floer homology we have used Lipshitz's cylindrical reformulation, by setting up the theory in  $\Sigma_{\max} \times [0, 1] \times \mathbb{R}$ . This was convenient because of the bubbling issues that may occur in the negatively monotone manifold  $\text{Sym}^{g-k}(\Sigma_{\max})$ . However, in the strongly negative case, when  $g < 2k$ , where bubbling can be ruled out for a generic path  $J_s$  of almost complex structures on  $\text{Sym}^{g-k}(\Sigma_{\max})$  on the grounds that the moduli space of bubbles in this case has negative virtual dimension. In fact, the proof of the lemma below shows that in this case, one can also use a path of integrable complex structures as in the case of the usual Heegaard Floer homology to make sense of this group. Thus, the Floer homology groups can be formulated as a Lagrangian intersection theory in  $\text{Sym}^{g-k}(\Sigma_{\max})$ .

**Lemma 24** *Suppose that  $Y$  admits a broken fibration with  $g < 2k$ . Then for  $\mathfrak{s} \in \mathcal{S}(Y|_{\Sigma_{\min}})$ ,*

$$QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq HF(\text{Sym}^{g-k}(\Sigma_{\max}); \alpha_1 \times \dots \times \alpha_{g-k}, \beta_1 \times \dots \times \beta_{g-k}, \mathfrak{s}; \Lambda)$$

*Proof.* We first argue that for a generic path of almost complex structures  $\{j_s\}$  on  $\Sigma_{\max}$  the induced integrable complex structures  $\text{Sym}^{g-k}(j_s)$  achieve transversality for the holomorphic disks mapping to  $\text{Sym}^{g-k}(\Sigma_{\max})$  which contribute to the differential and furthermore for these complex structures no bubbling can occur because of the strong negativity assumption  $g < 2k$ . The fact that these complex structures achieve transversality is standard and follows exactly as in the case of the usual Heegaard Floer homology set-up, see for example Proposition A.5 of [12]. To avoid bubbling, we make use of the Abel-Jacobi map:

$$AJ : \text{Sym}^{g-k}(\Sigma_{\max}) \rightarrow \text{Jac}(\Sigma_{\max})$$

The assumption  $g < 2k$  ensures that the Abel-Jacobi map is injective for  $j$  chosen outside of a subset of complex codimension at least 1 (so that for a generic path  $j_s$  it's injective for all  $s$ ). A generic choice of  $j_s$  therefore ensures that there cannot be any non-constant holomorphic spheres mapping to  $\text{Sym}^{g-k}(\Sigma_{\max})$ . One can also rule out disk bubbles in the same way: since the inclusions of  $\alpha_1 \times \dots \times \alpha_{g-k}$  and  $\beta_1 \times \dots \times \beta_{g-k}$  to  $\text{Sym}^{g-k}(\Sigma_{\max})$  are injective at the level of fundamental groups and since the Abel-Jacobi map is injective and induces an isomorphism on the first homology when  $g < 2k$ , the image of a holomorphic disc by the Abel-Jacobi map represents a trivial relative homology class, therefore it is trivial. Hence, there cannot be any non-constant holomorphic disk bubbles.

Now, applying the reformulation of Lipshitz, as in [12], allows us to translate the Lagrangian Floer homology in  $\text{Sym}^{g-k}(\Sigma_{\max})$  “tautologically” to the cylindrical set-up in  $\Sigma_{\max} \times [0, 1] \times \mathbb{R}$  (see appendix A in [12]).  $\square$

Finally, we are ready state our theorem that establishes the isomorphism between quilted Floer homology groups arising from Lagrangian correspondences with Heegaard Floer homology.

**Theorem 25** *Suppose that  $Y$  admits a broken fibration with  $g < 2k$  and furthermore assume that the widths of holomorphic quilts can be shrunk to zero without hitting any figure eight bubbles. Then for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{min})$ ,*

$$HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f; \mathfrak{s}, \Lambda) \simeq QFH(Y, f; \mathfrak{s}, \Lambda)$$

*Proof.* The proof will be obtained by putting together the results obtained so far together with Wehrheim-Woodward's Theorem 5.0.3 in [28] which allows one to compose Lagrangian correspondences without changing the Floer homology groups. More precisely, Theorem 19 and Lemma 24 give us that  $HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f; \mathfrak{s}, \Lambda) \simeq HF(\text{Sym}^{g-k}(\Sigma_{max}); \alpha_1 \times \dots \times \alpha_{g-k}, \beta_1 \times \dots \times \beta_{g-k}; \Lambda)$ . Now, Lemma 23 expresses the Lagrangians  $\alpha_1 \times \dots \times \alpha_{g-k}$  and  $\beta_1 \times \dots \times \beta_{g-k}$  as transverse and embedded compositions of the Lagrangians  $L_{\alpha_i}$  and  $L_{\beta_j}$ . Therefore, we are in a position to apply Wehrheim-Woodward theorem (for the statement of this theorem in the case we are using here, namely when the differential is perturbed by the intersection number with  $n_w$  see also Theorem 6.4 in [15]), which says that by shrinking the width of the strips which are part of the holomorphic quilts contributing to the differential one can obtain an isomorphism between the Floer homology of the Lagrangians  $\alpha_1 \times \dots \times \alpha_{g-k}, \beta_1 \times \dots \times \beta_{g-k}$  and the quilted Floer homology of the Lagrangian correspondences  $L_{\alpha_1}, \dots, L_{\alpha_{g-k}}$  and  $L_{\beta_1}, \dots, L_{\beta_{g-k}}$ . This completes the proof.  $\square$

Recall from [28] that a figure-eight bubble is given by a triple of holomorphic maps:

$$v_0 : \mathbb{R} \times (-\infty, -1] \rightarrow A, \quad v_1 : \mathbb{R} \times [-1, 1] \rightarrow B, \quad v_2 : \mathbb{R} \times [1, \infty) \rightarrow C$$

such that

$$(v_0(\tau, -1), v_1(\tau, -1)) \in L_{AB}, \quad (v_1(\tau, 1), v_2(\tau, 1)) \in L_{BC}$$

where  $A, B$  and  $C$  are symplectic manifolds, and  $L_{AB} \subset A \times B$  and  $L_{BC} \subset B \times C$  are Lagrangian correspondences. This is called a figure-eight bubble since, after compactifying the domain to  $\mathbb{C}P^1 = S^2$ , when viewed from  $z = \infty$  the lines  $\text{Im}(z) = \pm 1$  appear as a figure eight. It is conjectured in [28] that the maps  $(v_0, v_1, v_2)$  can be extended continuously to  $S^2$  by a point  $(v_0(\infty), v_1(\infty), v_2(\infty))$  that lies in both  $L_{AB} \times C$  and  $A \times L_{BC}$ . In the next lemma, we show that if we assume this removal of singularities at  $z = \infty$  for finite energy figure-eight bubbles, then figure-eight bubbles can be avoided in the proof of the previous theorem. We use the removal of singularity assumption in order to define the homotopy class of a figure-eight bubble which we then show to be trivial in the case  $g < 2k$ .

**Lemma 26** *Assuming the removal of singularities at  $z = \infty$  for figure-eight bubbles and  $g < 2k$ , the widths of the strips occurring in the differential of  $QFH(Y, f; \Lambda)$  can be shrunk to zero without*

hitting any figure-eight bubbles, i.e., the assumption about non-occurrence of figure-eight bubbles can be removed in Theorem 25.

*Proof.* We first explain how to associate a homotopy class with a figure-eight bubble. Consider the figure-eight bubble as two polar caps and an equatorial region, mapping to manifolds  $A, B, C$  ( $B$  is where the strip near the equator maps) - with seams mapping to the correspondences  $L_{AB}$  and  $L_{BC}$  in  $A \times B$  and in  $B \times C$ . The image of the equator is a loop  $\gamma$  inside  $B$ .

Let  $l_1$  be the loop obtained by reflecting the equator along the “north” seam: it’s a loop inside the northern polar region, bounding some disc  $D_1$ . Similarly, let  $l_2$  be the loop obtained by reflecting the equator along the “south” seam, i.e. a loop inside the southern polar region, bounding some disc  $D_2$ . All these loops and the equator touch each other at the point at infinity where the seams come together.

Now we can deform the maps in the polar regions so that they are constant over  $D_1$  and  $D_2$ : namely, let  $(a, b, c)$  in  $A \times B \times C$  be the value of the map at the point at infinity where everything attaches together (This is the precise moment where we assume that there is a removal of singularity theorem for the figure-eight bubbles). Then after a homotopy we can assume that every point of  $D_1$  maps to  $a$ , and every point of  $D_2$  maps to  $c$ . After we do this, cut the domain along the equator, and look first at the north hemisphere. We have on one hand a strip in  $B$ , and on the other hand a disc in  $A$ , but the disc is constant north of  $l_1$ , so we can cut it to a strip in  $A$ . Then we can reflect, and get a strip in  $A \times B$ , with one boundary (the seam) on the given Lagrangian correspondence  $L_{AB}$ , and the other boundary (the equator and the reflected loop  $l_1$ ) mapping to  $\{a\} \times \gamma$ . Similarly from the southern hemisphere we get a strip in  $B \times C$ , with one boundary mapping on the given Lagrangian correspondence  $L_{BC}$ , and the other boundary mapping to  $\gamma \times \{c\}$ . Now take the strip in  $A \times B$ , and make it into a map to  $A \times B \times C$  just by taking the constant function  $c$  in the last factor: so we get a strip with boundaries in  $L_{AB} \times \{c\}$  and  $\{a\} \times \gamma \times \{c\}$ . Take the strip in  $B \times C$  and similarly add in the constant map to  $a$  in  $A$  to get a strip with boundaries in  $\{a\} \times L_{BC}$  and  $\{a\} \times \gamma \times \{c\}$ . Now we can glue these two together and get a strip in  $A \times B \times C$  with boundaries in  $L_{AB} \times \{c\}$  and  $\{a\} \times L_{BC}$ . So there is a relative homology class associated with it.

We next argue that in the case  $g < 2k$ , this homology class has to be trivial for any figure-eight bubble that might arise from shrinking the width of the strips that are considered in the definition of  $QFH(Y, f, \mathfrak{s}; \Lambda)$  in the above theorem; this implies that figure-eight bubbles would be just constant maps, i.e. no such bubbling occurs. Namely, suppose  $A = \text{Sym}^n(\Sigma)$ ,  $B = \text{Sym}^{n-1}(\Sigma_L)$  and  $C = \{pt\}$ , where  $L$  is an  $\alpha$  curve or a  $\beta$  curve. (Recall that  $\Sigma_L$  is obtained from  $\Sigma$  by collapsing  $L$  to a point and taking the normalisation). This case is sufficient for our purposes as, when we shrink the widths of holomorphic strips that arise in quilted Floer homology, we can always do successive shrinking in an order such that one of the seam conditions involves the zeroth symmetric of the lowest genus fibre which is just a point. In our case we have  $n \leq g - k$  which is assumed to be less than  $k$ , and the genus of  $\Sigma$  is  $k + n$  which is greater than  $2n$ . As in Lemma 24, these conditions

guarantee that the Abel-Jacobi maps from  $A$  and  $B$  to corresponding Picard tori are injective. Furthermore, the Abel-Jacobi map is holomorphic when it is viewed as a map from the relative Hilbert scheme to the relative Picard fibration. Therefore, a figure-eight bubble for the symmetric products gives rise to a figure-eight bubble for the Picard tori, with seam conditions given by taking the images of  $L_{AB}$  and  $L_{BC}$  by the Abel-Jacobi map. Let us denote by  $A'$  and  $B'$  be the Picard tori  $T^{2(k+n)}(\Sigma)$ ,  $T^{2(k+n-1)}(\Sigma_L)$ , and  $C' = \{pt\}$  so that the Abel-Jacobi map sends  $A$ ,  $B$  to  $A'$ ,  $B'$ , and let  $L_{A'B'}$  and  $L_{B'C'}$  be the images of the Lagrangian correspondence  $L_{AB}$  and  $L_{BC}$  by the Abel-Jacobi map. Now, in the previous paragraph, we have seen that homotopically a figure-eight bubble can be regarded as a loop based at the constant path at  $(a, b, c)$  in the path space  $\Omega(\{a\} \times L_{BC}, L_{AB} \times \{c\})$ , i.e., the set of paths  $\omega : [0, 1] \rightarrow A \times B \times C$  such that  $\omega(0) \in \{a\} \times L_{BC}$  and  $\omega(1) \in L_{AB} \times \{c\}$ . Thus the homotopy class of a figure-eight bubble is an element in  $\pi_1(\Omega(\{a\} \times L_{BC}, L_{AB} \times \{c\}))$ . Now note that, the evaluation maps  $(ev_0, ev_1) : \Omega(\{a\} \times L_{BC}, L_{AB} \times \{c\}) \rightarrow (\{a\} \times L_{BC}) \times (L_{AB} \times \{c\})$  give rise to a Serre fibration with fibre space homotopy equivalent to the loop space  $\Omega(A \times B \times C)$ :

$$\begin{array}{ccc} \Omega(A \times B \times C) & \longrightarrow & \Omega(\{a\} \times L_{BC}, L_{AB} \times \{c\}) \\ & & \downarrow \\ & & (\{a\} \times L_{BC}) \times (L_{AB} \times \{c\}) \end{array}$$

A similar argument applies to  $A'$ ,  $B'$  and  $C'$  with the seam conditions  $L_{A'B'}$  and  $L_{B'C'}$ . Therefore, we get the following homotopy exact sequences which are connected by the Abel-Jacobi maps:

$$\begin{array}{ccccc} \pi_1(\Omega(\{a\} \times L_{BC}, L_{AB} \times \{c\})) & \longrightarrow & \pi_1(L_{BC}) \times \pi_1(L_{AB}) & \longrightarrow & \pi_1(A \times B \times C) \\ & & \downarrow (aj)_* & & \downarrow \\ 0 & \longrightarrow & \pi_1(\Omega(\{a'\} \times L_{B'C'}, L_{A'B'} \times \{c'\})) & \longrightarrow & \pi_1(L_{B'C'}) \times \pi_1(L_{A'B'}) \end{array}$$

Now, we would like to show that the image of a figure-eight bubble by the Abel-Jacobi map is contractible. Thus, we would like to show that the vertical map denoted by  $(aj)_*$  is zero. In fact, when  $n > 2$  one can show that  $\pi_1(\Omega(\{a'\} \times L_{B'C'}, L_{A'B'} \times \{c'\}))$  is zero, but in order to also cover the case  $n = 2$ , we need to do a little bit of diagram chasing. First observe that the rightmost vertical arrow is an isomorphism since  $L_{A'B'} = AJ(L_{AB})$ ,  $L_{B'C'} = AJ(L_{BC})$  and the Abel-Jacobi map is injective. Also note that  $\pi_2(A' \times B' \times C') = 0$  since  $A'$  and  $B'$  are tori and  $C' = \{pt\}$ . Therefore, it suffices to show that the map  $\pi_1(L_{BC}) \times \pi_1(L_{AB}) \rightarrow \pi_1(A \times B \times C)$  is injective. Perutz shows in [23] Lemma 3.18 that the inclusion of a Lagrangian correspondences is injective at the level of fundamental groups. This is done by calculating the maps  $\pi_1(L_{AB}) \rightarrow \pi_1(A \times B)$  and  $\pi_1(L_{BC}) \rightarrow \pi_1(B \times C)$ . In the case at hand topologically we have  $A = \text{Sym}^n(\Sigma)$ ,  $B = \text{Sym}^{n-1}(\Sigma_L)$ ,  $L_{AB}$  is a trivial circle bundle over  $\text{Sym}^{n-1}(\Sigma_L)$  so has fundamental group  $\mathbb{Z} \times H_1(\Sigma_L)$  when  $n > 2$  and  $\mathbb{Z} \times \pi_1(\Sigma_L)$  when  $n = 2$ , where the fibre of the circle bundle generates the  $\mathbb{Z}$  component. This maps to  $\pi_1(A \times B) = H_1(\Sigma) \times H_1(\Sigma_L)$  when  $n > 2$  and  $\pi_1(A \times B) = H_1(\Sigma) \times \pi_1(\Sigma_L)$  when  $n = 2$ , where the fibre class is mapped to  $L$  and the restriction to  $H_1(\Sigma_L)$  sends a class  $[\gamma]$  to

$([\gamma], [\gamma])$  in  $H_1(\Sigma) \times H_1(\Sigma_L)$ , where the first component of this map is given by making sure  $\gamma$  doesn't pass through the region of normalization and hence can be identified with a loop in  $\Sigma$  via parallel transport. Furthermore,  $C = \{pt\}$  and  $L_{BC}$  is a product torus  $L_1 \times \dots \times L_{n-1}$ , such that when viewed as loops on  $\Sigma$ ,  $L, L_1, \dots, L_{n-1}$  are linearly independent in  $H_1(\Sigma)$ .

Thus, the map  $\pi_1(L_{BC}) \times \pi_1(L_{AB}) \rightarrow \pi_1(A \times B \times C)$  is given by:

$$\begin{aligned} \mathbb{Z} \times H_1(\Sigma_L) \times \mathbb{Z}^{n-1} &\rightarrow H_1(\Sigma) \times H_1(\Sigma_L) \\ (m, \gamma, r_1, \dots, r_{n-1}) &\rightarrow (mL + \gamma, \gamma + r_1L_1 + \dots + r_{n-1}L_{n-1}) \end{aligned}$$

for  $n > 2$ . The formula is similar in the case  $n = 2$  if one replaces  $H_1(\Sigma_L)$  by  $\pi_1(\Sigma_L)$ . This map is clearly injective since the lift of  $\gamma$  to  $\Sigma$  and  $L$  are independent in  $H_1(\Sigma)$ . Therefore, the map  $(aj)_*$  is zero, and the image of a figure-eight bubble by the Abel-Jacobi map has to be contractible, which implies that it cannot carry any symplectic energy, thus it cannot have any holomorphic representative unless it is a constant map. Since the Abel-Jacobi map is holomorphic, this completes the proof of the non-occurrence of figure-eight bubbles.  $\square$

## 4.2 Floer's excision theorem

Here we describe a proof of Floer's excision theorem for quilted Floer homology. In light of the theorem in the previous section this gives a new and more straightforward proof of Floer's excision theorem for Heegaard Floer homology. In this section, we will denote by  $QFH(Y, f; \Lambda)$ , the quilted Floer homology of  $(Y, f)$  thought as the Floer homology group,  $HF(\mathbb{L}_\alpha, \mathbb{L}_\beta; \mathfrak{s}, \Lambda)$ , using Theorem 25.

In order to prepare the set-up, consider two broken fibrations  $f_1 : Y_1 \rightarrow S^1$  and  $f_2 : Y_2 \rightarrow S^1$ . Let  $g_1$  and  $g_2$  be the genera of the maximal genus fibres of  $f_1$  and  $f_2$  and  $k$  be the genus of the minimal genus fibres of  $f_1$  and  $f_2$  which we assume to be equal. Let us denote by  $\Sigma_i$  a fixed minimal genus fibre of  $f_i$ . Now cut each  $Y_i$  along  $\Sigma_i$ , to obtain manifolds  $Y'_i$  with boundary  $\Sigma_i \cup (-\Sigma_i)$ . Choose a diffeomorphism  $\phi : \Sigma_1 \rightarrow \Sigma_2$  and form a new closed 3-manifold  $Y$  by gluing  $\Sigma_1$  to  $-\Sigma_2$  and  $-\Sigma_1$  to  $\Sigma_2$  using  $\phi$ .  $Y$  comes equipped with a broken fibration  $f$  induced by  $f_1$  and  $f_2$ . Furthermore, given  $\text{spin}^c$  structures  $\mathfrak{s}_1 \in \mathcal{S}(Y_1|\Sigma_1)$  and  $\mathfrak{s}_2 \in \mathcal{S}(Y_2|\Sigma_2)$ , one gets an induced  $\text{spin}^c$  structure  $\mathfrak{s} = \mathfrak{s}_1 \# \mathfrak{s}_2 \in \mathcal{S}(Y|\Sigma_{\min})$ . Floer's excision theorem in this context is as follows :

**Theorem 27** *Suppose that  $g_1 + g_2 < 3k$ , then for  $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$  we have a split short exact sequence of graded groups*

$$\begin{aligned} \bigoplus_{\{\mathfrak{s}_1, \mathfrak{s}_2 | \mathfrak{s}_1 \# \mathfrak{s}_2 = \mathfrak{s}\}} QFH(Y_1, f_1, \mathfrak{s}_1; \Lambda) \otimes QFH(Y_2, f_2, \mathfrak{s}_2; \Lambda) &\rightarrow QFH(Y, f, \mathfrak{s}; \Lambda) \\ \rightarrow \bigoplus_{\{\mathfrak{s}_1, \mathfrak{s}_2 | \mathfrak{s}_1 \# \mathfrak{s}_2 = \mathfrak{s}\}} \text{Tor}(QFH(Y_1, f_1, \mathfrak{s}_1; \Lambda), QFH(Y_2, f_2, \mathfrak{s}_2; \Lambda)) & \end{aligned}$$

*Proof.* The proof of this theorem in the setting of  $QFH(Y, f; \Lambda)$  follows from the definition of quilted Floer homology in a straightforward way. The crucial observation is that for the  $\text{spin}^c$  structures in consideration, the function  $\nu : S^1 \setminus \text{crit}(f) \rightarrow \mathbb{Z}_{\geq 0}$  defined by  $\langle c_1(\mathfrak{s}), F_s \rangle = 2\nu(s) + \chi(F_s)$  is zero for the fibres  $\Sigma_1$ ,  $\Sigma_2$  and  $\Sigma_{\min}$ , hence the cutting and gluing operations take place where the holomorphic quilts contributing to differentials live in a zeroth symmetric product and so are constant in those regions. Therefore there is an isomorphism of the tensor product of chain complexes associated with  $(Y_1, f_1)$  and  $(Y_2, f_2)$  with the chain complex associated with  $(Y, f)$ , thus the theorem follows.  $\square$

One issue that we are not addressing here is the fact that, as constructed,  $f$  does not satisfy the conditions stated at the beginning of Section 2.1 on the ordering of index 1 and 2 critical values. However the results of [11] imply that perturbing  $f$  to achieve these conditions would not affect  $QFH(Y, f, \mathfrak{s})$ .

We remark that the excision theorem for quilted Floer homology allows us to apply the constructions developed by Kronheimer and Mrowka in [7] in the context of quilted Floer homology. In particular, one can define knot invariants in this way.

### 4.3 4-manifold invariants

We first recall the definition of broken Lefschetz fibrations on smooth 4-manifolds.

**Definition 28** *A broken fibration on a closed 4-manifold  $X$  is a smooth map to a closed surface with singular set  $A \cup B$ , where  $A$  is a finite set of singularities of Lefschetz type near which a local model in oriented charts is the complex map  $(w, z) \rightarrow w^2 + z^2$ , and  $B$  is a 1-dimensional submanifold along which the fibration is locally modelled by the real map  $(t, x, y, z) \rightarrow (t, x^2 + y^2 - z^2)$ ,  $B$  corresponding to  $t = 0$ .*

It was proven in [9] that every closed oriented smooth 4-manifold admits an *equatorial* broken Lefschetz fibration to  $S^2$  (see also [2] where the authors give a new proof of this result using Kirby calculus). Equatorial here means that the 1-dimensional part of the critical value set is a set of embedded parallel circles on  $S^2$ . Lagrangian matching invariants of a 4-manifold as defined by Perutz in [23] are obtained by counting quilted holomorphic sections of a broken fibration associated with the 4-manifold. These invariants, which are conjecturally equal to Seiberg-Witten invariants, have a TQFT-like structure where the three manifold invariants are the quilted Floer homology groups that we have discussed in this paper. Similarly, Heegaard Floer homology is the three manifold part of a TQFT-like structure, which underlies the construction of Ozvath-Szabo 4-manifold invariants [20].

By cutting a broken fibration along a family of circles that are transverse to the equatorial circles of critical values, one can obtain a cobordism decomposition of the 4-manifold, such that each

cobordism is an elementary cobordism, namely it is a cobordism obtained by either a one or two handle attachment. Therefore, because of Theorem 25, in order to equate the above mentioned four-manifold invariants for the  $\text{spin}^c$  structures which satisfy the adjunction equality with respect to the minimal genus fibre of the broken fibration, one needs to check only that the cobordism maps for one and two handle attachments in both theories coincide. This will be in turn obtained by extending the techniques developed in this paper to cobordism maps. We plan to investigate this latter claim in a sequel to this paper. This will in particular prove that for the  $\text{spin}^c$  structures considered, the Lagrangian matching invariants are independent of the broken fibration that is chosen on the 4-manifold.

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