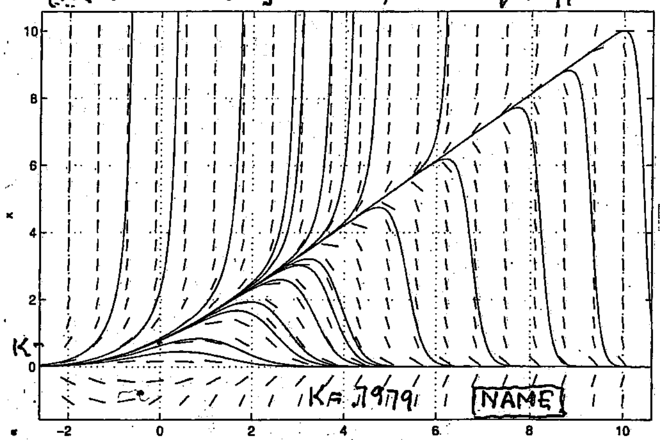
18.03 Prob. set 3 Solutions – Spr. 2007

1 a) $K = .7978$ or $.7979$

($x(0) = .7978$ goes down; $.7979$ goes up)



b) $\boxed{x' = -tx + x^2}$ Bernoulli;

$$\frac{x'}{x^2} = \frac{-t}{x} + 1 \qquad \begin{cases} v = 1/x \\ v' = -x'/x^2 \end{cases}$$

$\boxed{v' - tv = -1}$ Linear

Int. factor: $e^{-t^2/2}$

$$(ve^{-t^2/2})' = -e^{-t^2/2}$$

$$ve^{-t^2/2} = -\int_0^t e^{-u^2/2}\,du + c$$

$$v = e^{t^2/2}\left[c - \int_0^t e^{-u^2/2}\,du\right];$$

$$v(0) = \frac{1}{x(0)} = \frac{1}{x_0} = 1[c - 0] = c$$

$$\frac{1}{v} = \boxed{x = \frac{e^{-t^2/2}}{\frac{1}{x_0} - \int_0^t e^{-u^2/2}\,du}} \quad (*)$$

c) $\displaystyle\int_0^\infty e^{-u^2/2}\,du = \int_0^\infty e^{-v^2}\,dv \cdot \sqrt{2} = \frac{\sqrt{\pi}}{\sqrt{2}}$ $\quad u = v\sqrt{2}$

$\therefore \int_0^t e^{-u^2/2}\,du$ increases to the limit $\sqrt{\frac{\pi}{2}}$.

$\therefore$ Looking at (*): $\frac{1}{x_0} > \sqrt{\frac{\pi}{2}} \Rightarrow$ denom. never 0, $\therefore \lim_{t\to\infty} x(t) = 0$

while, $(\therefore x_0 < \sqrt{\frac{2}{\pi}})$

$\frac{1}{x_0} < \sqrt{\frac{\pi}{2}}$ $(\therefore x_0 > \sqrt{2/\pi}) \Rightarrow$ denom. $= 0$ for some point $t_1 > 0$, $\lim_{t\to t_1} x(t) = \infty$

Hence $K = \sqrt{\frac{2}{\pi}} \approx .7979$

d) If $x_0 = K$, $\frac{1}{x_0} = \sqrt{\frac{\pi}{2}}$ & (*) is of form $\frac{0}{0}$

By L'Hosp: $\displaystyle\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{e^{-t^2/2}(-t)}{-e^{-t^2/2}} = \infty.$

2 $B = .13 + \frac{2}{3}(.01N)$ $\quad N = $ rec. # $1 \to 33$

$\therefore B = .14 \to .35$

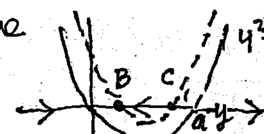
(Everyone in recit'n should have same value of B)

a) answer should be $a = B + \frac{1}{4B}$ (for the above value of B)

b) answer is $a = 1$ (for all recit'ns)

c) <u>Analytic derivation of answer (a):</u>

Picture



$y^2 - ay$ $\quad$ --- shows $y^2 - ay + 1/4$

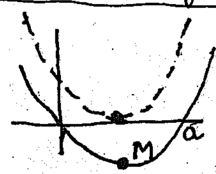
want to choose a so that <u>lower</u> (stable) root of $y^2 - ay + 1/4$ has value B.

$\therefore B^2 - aB + 1/4 = 0$,

so $\quad a = \dfrac{B^2 + .25}{B} = B + \dfrac{1}{4B}$

<u>Analytic deriv'n of answer (b):</u>

Picture:



$y^2 - ay = f(y)$

--- $y^2 - ay + 1/4$

What value of a gives this picture?

The pt. of tangency is $a/2$ (by symmetry)

The pt. $M = (a/2, f(a/2))$ $\quad f(a/2) = \frac{a^2}{4} - \frac{a^2}{2}$

$= (a/2, -a^2/4)$

$\therefore$ want

$$\underset{\text{dashed curve}}{y^2 - ay + 1/4} = \underset{\text{solid curve moved up to dashed position.}}{y^2 - ay + \frac{a^2}{4}}$$

so $a^2 = 1$

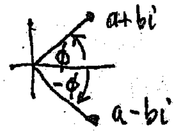
$a = 1$

3 a) Multiplying in Cartesian form:

$(a-bi)(\cos\omega t + i\sin\omega t)$ ⊛

$= \boxed{a\cos\omega t + b\sin\omega t}$ ← REAL PART

$\quad + i(a\sin\omega t - b\cos\omega t)$



$r = \sqrt{a^2+b^2}$

$a - bi = re^{-i\phi}$

$\cos\omega t + i\sin\omega t = e^{i\omega t}$

$\therefore$ ⊛ $= re^{i(\omega t-\phi)}$

$= \sqrt{a^2+b^2}\left(\cos(\omega t-\phi) + i\sin(\omega t-\phi)\right)$

Real part $= \boxed{\sqrt{a^2+b^2}\cos(\omega t-\phi)}$

Equating this with ⊛ real part gives the sinusoidal identity.

b) $e^{3i\theta} = \cos 3\theta + i\sin 3\theta$ (1)

$= (\cos\theta + i\sin\theta)^3$ (2)

$= \cos^3\theta + 3i\cos^2\theta\sin\theta$
$\quad + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$

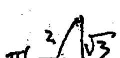
Equating the real parts:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$ ⊛

If $\theta = \pi/9$, $3\theta = \pi/3$, $\boxed{\cos 3\theta = 1/2}$



$\therefore$ by ⊛: if $x = \cos\theta$, $\theta = \pi/9$

$\boxed{4x^3 - 3x - 1/2 = 0}$ has $\cos\pi/9$ as a root

[It can't be a rational number since this equation doesn't factor (using rational coefficients)].

4 Solve $x^4 - 4x^2 + 16 = 0$

a) set $u = x^2$: $u^2 - 4x + 16 = 0$

By quad. formula: $u = \dfrac{4 \pm \sqrt{16 - 4\cdot 16}}{2}$

$\therefore u = 2(1 \pm \sqrt{-3}) = 2(1 \pm i\sqrt{3})$
$= 2\cdot 2e^{\pm i\pi/3}$

b) ..

In polar form:

$u = x^2 = 4e^{i\pi/3}$

$x = \begin{cases} 2e^{i\pi/6} = 2(\sqrt{3}+i)/2 \\ 2e^{i(\frac{\pi}{6}+\pi)} = -2(\sqrt{3}+i)/2 \end{cases}$

$u = x^2 = 4e^{-i\pi/3}$

$x = \begin{cases} 2e^{-i\pi/6} = 2\left(\frac{\sqrt{3}-i}{2}\right) \\ 2e^{-i\pi/6 + i\pi} \\ 2e^{i5\pi/6} = -2\left(\frac{\sqrt{3}-i}{2}\right) \end{cases}$

$\therefore$ 4 roots are

$\pm(\sqrt{3} \pm i)$ cartesian

$\pm 2e^{\pm i\pi/6}$ polar