18.03 Prob Set 2 solutions

1 $y' = y$

a) Values of $y(3.00)$ for the solution $y = .10e^x$, via Euler

$h = 5$ $\quad y(3.00) = 1.14$

$h = .25$ $\quad y(3.00) = 1.46$

actual $\quad y(3.00) = 1.98$

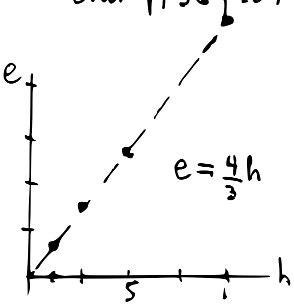
correct $\quad y(3.00) = 2.00 \leftarrow$ (to nearest even dec place)

Discrepancies come from uncertainties in arrow placement

b) $y' = (\sin x)y$ Calculations shown for starting value $\boxed{y(-4) = 1.50}$

Table:

h	1.0	.5	.25	.125	actual
y(4)	.10	.82	1.14	1.32	1.46
error	1.36	.64	.32	.14	0



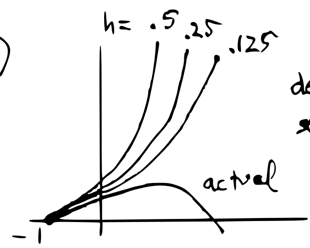
For two dec place accuracy, want

$e = .005$

$.005 = \frac{4}{3} h$

$h = .003$ or $.004$

c)



soln behavior at ∞ depends on $y(0)$ very sensitively – the 4 Euler approximations are all too high at $y(0)$

2 $y(a) = y_0 + \int_0^a f(x)\,dx$

Using trapezoidal rule for $\int_0^a f(x)\,dx$

$$y(a) \approx y_0 + h\left[\frac{f(0)}{2} + f(x_1) + \quad + f(x_{n-1}) + \frac{f(a)}{2}\right]$$

(where $x_1 = h$, $x_2 = 2h$, $\quad x_{n-1} = (n-1)h$)

Using Improved Euler for $y' = F(x,y)$, where $F(x,y) = f(x)$

$y_{k+1} = y_k + \frac{h}{2}(A_k + B_k) \leftarrow A_k = F(x_k, y_k) = f(x_k)$

$B_k = F(x_{k+1}, \tilde{y}_{k+1}) = f(x_{k+1})$

$y_1 = y_0 + \frac{h}{2}(f(0) + f(x_1))$

$y_2 = y_1 + \frac{h}{2}(f(x_1) + f(x_2))$

$\leftarrow x_n = a$, note

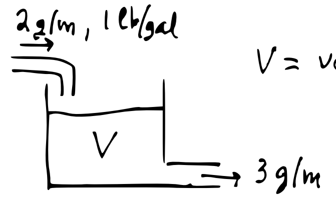
$y_n = y_{n-1} + \frac{h}{2}(f(x_{n-1}) + f(x_n))$

$\boxed{\text{Add the equations; cancel } y_1, \; , y_{n-1}}$

$$y(a) \approx y_n = y_0 + \frac{h}{2}\left(f(0) + 2f(x_1) + \quad + 2f(x_{n-1}) + f(a)\right)$$

this agrees with the trapezoidal rule approximation

3 a)



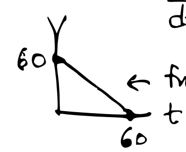
V = volume of brine at time t (m = minutes, g = gallons)

$x(t)$ = salt in tank at time t (lb/gal)

The ODE $\quad \frac{dx}{dt} = \text{rate of salt inflow} - \text{rate of salt outflow}$

$= 2 \cdot 1 - 3 \cdot \frac{x}{V}$ $\quad$ [flow rate · salt conc]

$\therefore \frac{dx}{dt} + \frac{3x}{V} = 2$



$\leftarrow$ from the data, $\quad V = 60 - t$

so ODE is: $\quad \boxed{\frac{dx}{dt} + \frac{3x}{60-t} = 2}$

Integrating factor: $\quad e^{-3\ln(60-t)} = (60-t)^{-3}$

Multiplying through by this

$$[x(60-t)^{-3}]' = 2(60-t)^{-3}$$

$$\frac{x}{(60-t)^3} = \frac{1}{(60-t)^2} + c \qquad \text{[don't forget the chain rule!]}$$

$x(0) = 0 \Rightarrow c = -\frac{1}{60^2}$

$$\therefore \boxed{x = (60-t) - \frac{1}{60^2}(60-t)^3}$$

b) To find x_m, maximum salt, need time t_m of max salt

$$\frac{dx}{dt} = -1 + \frac{3}{60^2}(60-t)^2$$

this $= 0$ if $\quad (60-t)^2 = \frac{60^2}{3}$

$60 - t = \pm\frac{60}{\sqrt{3}} = \pm 20\sqrt{3}$

$\therefore t_m = 60 - 20\sqrt{3}$ $\quad$ (other value is outside $[0, 60]$)

Amt of salt in tank at t_m $\quad 20\sqrt{3} - \frac{1}{60^2}(20\sqrt{3})^3$

$= 20\sqrt{3} - 20\frac{\sqrt{3}}{3} = \frac{40}{3}\sqrt{3}$ lbs salt ≈ 23.1 lbs

4 $T(t)$ = egg temp °C, at time t (min)

a) $T_{ext} = 100e^{-at}$ (water-bath)

$T(0) = 0$ (given)

Newton $\frac{dT}{dt} = k(T_{ext} - T)$

$$\therefore \boxed{\frac{dT}{dt} + kT = 100k e^{-at}}$$

Multiply through by e^{kt} (integrating factor)

$$(Te^{kt})' = 100k\, e^{(k-a)t}$$

$$(*) \quad Te^{kt} = \frac{100k}{k-a} e^{(k-a)t} + c$$

$$T(0) = 0 \Rightarrow c = -\frac{100k}{k-a} = -\frac{100(1)}{1-.05} = -200$$

$$\therefore \boxed{T = 200(e^{-at} - e^{-kt})} \quad \text{(from } *)$$

b) Max temperature: when $\frac{dT}{dt} = 0$ (t_m)

$$0 = \frac{dT}{dt} = 200(-ae^{-at} + ke^{-kt})$$

solving for t_m $\quad ke^{-kt_m} = ae^{-at_m}$

$$\frac{k}{a} = e^{(k-a)t_m} \qquad k = 1, \; a = .05$$

$$(**) \quad 2 = e^{.05 t_m} \qquad \ln 2 = .69$$

$$.69 = .05t$$

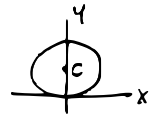
$$t_m = \frac{.69}{.05} \approx 14 \text{ mins}$$

To find $T(t_m)$, use (**)

$$T(t_m) = 200(e^{-.05t_m} - e^{-.1t_m})$$

$$= 200\left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right)$$

$$= 200 \cdot \frac{1}{4} = 50°C$$

5 a) 

Family (*)

$$x^2 + (y-c)^2 = c^2$$

Expanding: $x^2 + y^2 - 2cy = 0$

$$\therefore c = \frac{x^2+y^2}{2y}, \qquad c - y = \frac{x^2-y^2}{2y} \text{ (will need)}$$

To get ODE of family, diff (*)

$$2x + 2(y-c)y' = 0$$

$$\therefore y' = \frac{x}{c-y} = \boxed{\frac{2xy}{x^2-y^2} = y'}$$

b) Orthog. trajectories have ODE

$$y' = \frac{y^2-x^2}{2xy} \quad \text{(negative reciprocal of)}$$

$$y' = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right) \quad \text{homogeneous ODE}$$

Put $z = \frac{y}{x}$, $\quad xz = y$, $\quad z + xz' = y'$

$$z + x\frac{dz}{dx} = \frac{1}{2}\left(z - \frac{1}{z}\right)$$

$$x\frac{dz}{dx} = -\frac{1}{2}\left(z + \frac{1}{z}\right) = -\frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

Separating variables:

$$\frac{2z}{z^2+1}dz = -\frac{dx}{x}$$

$$\ln(z^2+1) = \ln(1/x) + c_1$$

exponentiating $(C = e^{c_1})$

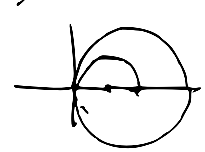
$$z^2 + 1 = C\frac{1}{x} \qquad z = y/x$$

$$\frac{y^2}{x^2} + 1 = \frac{C}{x} \qquad \text{mult by } x^2$$

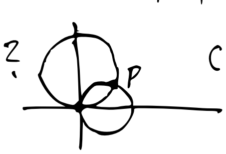
$$\boxed{x^2 + y^2 - cx = 0} \quad \text{Eq'n of family}$$

Complete square to help recognize it

$$\left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$$

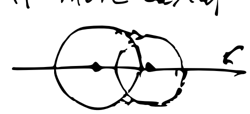


circles tan to y-axis at (0,0)

why? 

Circles are ⊥ at (0,0) must also be ⊥ at P

— see it more easily if you draw them



← line through the two centers

— circles are symmetric about this line