18 03 Problem Set 1 Solutions

1 $\frac{dv}{dt} = 24 - 02v - 01v^2$

a) $\frac{100dv}{V^2+2V-24} = -dt$

Use partial fractions

$$\frac{100}{(V-4)(V+6)} = \frac{10}{V-4} - \frac{10}{V+6}$$

Integrating, and combining ln

$$10 \ln \frac{V-4}{V+6} = -t + c_1$$

Exponentiate $(C = e^{c_1})$

$$\frac{V-4}{V+6} = Ce^{-t/10}$$

Put in initial condition $V=0, t=0$ $\Rightarrow C=-2/3$

Solve for V
(helps to write $-2/3\, e^{-t/10} = A$)
(temporarily)

$$V = \frac{4+6A}{1-A} = \boxed{\frac{4-4e^{-t/10}}{1+\frac{2}{3}e^{-t/10}}}$$

b) as $t \to \infty$, $v \to 4$ m/s
(cruising speed)

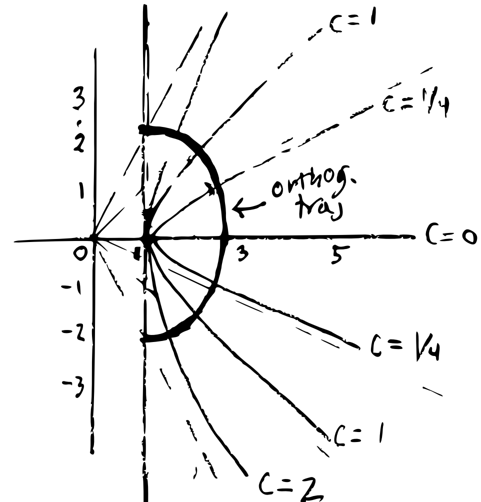
$$|v-4| = \left|\frac{4-4e^{-t/10} - 4 - 4\frac{2}{3}e^{-t/10}}{1+2/3\,e^{-t/10}}\right|$$

$< \frac{20}{3} e^{-t/10}$ (ignoring bottom which is $\gtrapprox 1$)
$\frac{20}{3} \approx 6\ 67$

$6\ 67 e^{-t/10} < 03$
if $e^{-t/10} < 005$
which is true if $\boxed{t = 50}$

2

a)



b) $cx^2 - y^2 = c \Rightarrow c = \frac{y^2}{x^2-1}$

Differentiate

$$2cx - 2yy' = 0 \Rightarrow y' = \frac{cx}{y} = \frac{y^2}{x^2-1}\frac{x}{y}$$

$\therefore$ ODE of family is:

$y' = \frac{yx}{x^2-1}$ orthog traj ODE $\boxed{\frac{dy}{dx} = \frac{1-x^2}{yx}}$

Solving:

$ydy = \left(\frac{1}{x} - x\right)dx \Rightarrow \frac{y^2}{2} = \ln x - \frac{x^2}{2} + c$

or $\boxed{x^2 + y^2 = 2\ln x + C}$ ORTHOG. TRAJS.

c) curve through $(e,0)$

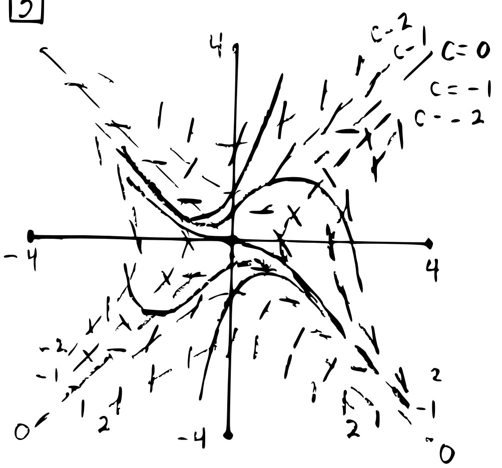
$e^2 + 0^2 = 2\cdot 1 + C$ $\quad C = e^2 - 2 \approx 5\ 3$

Intersects $x=1$

$1^2 + y^2 = 0 + 5.3$; $y^2 = 4.3$
$y \approx \pm 2.1$

(shown in picture)

3



(i) $.67 \approx K$ ($y(0) = 68$ goes up, $y(0) = 66$ goes down)
or start at $(4,4)$ — which goes down, + trace solution back to .66 approx)

(ii) $y(x)$ follows the isocline $y^2 - x^2 = -1$ closely, as x gets large, so $y(10) \approx \sqrt{10^2-1} = -\sqrt{99} \approx \boxed{.95}$

[It is trapped between $y^2 - x^2 = -1$ and $y^2 - x^2 = -2$
so $-\sqrt{99} < y(10) < -\sqrt{98}$
$-95 \lessapprox y(10) < -.90$ so $y = .97 \pm .02$ is safe

(iii) There is no such M, since the solution $y(x)$ through any (M_0, M_0) has a maximum pt at (M_0, M_0) (slope is 0 there) and $\therefore y(0) < K$.