18.03 Exam 1 Solutions Spring 2007

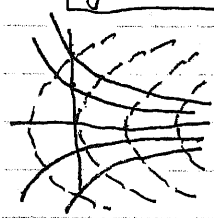
1 $y = ce^{-x} \Rightarrow y' = -ce^{-x}$

$\therefore \boxed{y' = -y}$ ODE for family

$\frac{dy}{dx} = \frac{1}{y}$ ODE for orthog. traject.

Solution: $y\,dy = dx \Rightarrow \frac{1}{2}y^2 = x + c_1$

$\therefore \boxed{y^2 = 2x + c}$ (parabolas)



original family in solid lines

orthog traj: $- - -$ lines

(backwards – sorry)

2 $y' = \frac{2}{x}y - \frac{y^2}{x}$ Bernoulli ODE

a) $\frac{y'}{y^2} = \frac{2}{x}\cdot\frac{1}{y} - \frac{1}{x}$ Put $u = \frac{1}{y}$

$\therefore u' = -y'/y^2$

$-u' = \frac{2}{x}u - \frac{1}{x}$ $y(1) = \frac{1}{2}$

or: $u' + \frac{2}{x}u = \frac{1}{x}$ $\Rightarrow u(1) = \frac{1}{y(1)} = 2$

b) Solution: $e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

Mult. by integrating factor:

$x^2u' + 2xu = x \Rightarrow (x^2u)' = x$

$\therefore x^2u = \frac{1}{2}x^2 + c$ | $u(1) = 2$: $2 = \frac{1}{2} + c$, $c = 3/2$

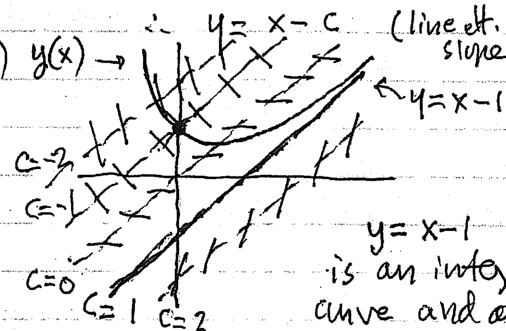
$\therefore u = \frac{1}{2} + \frac{3}{2x^2}$

$y = \frac{1}{\frac{x^2+3}{2x^2}} = \frac{2x^2}{x^2+3}$

3 a) Isoclines are $x - y = c$ (slope c)

$\therefore y = x - c$ (line ctr. has slope c)

b) $y(x) \rightarrow$



$y = x - 1$ is an integral curve and an isocline

c) $y(25) \approx x - 1\Big|_{x=25} = 24$

d)

n	x	y	x−y	h(x−y)
0	0	1	−1	−.1
1	.1	.9	−.8	−.08
2	.2	.82		

$y(.2) \approx .82$

answer too low, since soln is convex

(from the picture, or from

$y'' = 1 - y'$ $\therefore y''(0) = 1 - (-1) = 2 > 0$

$y' = x - y$

4 a) $\frac{dT}{dt} = k(T_e - T) = k(100e^{-kt} - T)$

$\therefore T' + kT = 100ke^{-kt}$

b) Int factor is e^{kt}:

$(e^{kt}T)' = 100k \Rightarrow e^{kt}T = 100kt + c$

$T(0) = 0 \Rightarrow c = 0$

$\therefore T = 100kte^{-kt}$

c) Max: $T' = 100k(e^{-kt} - kte^{-kt})$

$= 0$ if $1 = kt$, $\boxed{t = 1/k}$

Max $T = 100k\cdot\frac{1}{k}\cdot e^{-1} = 100e^{-1}$

$\approx \frac{100}{2.7} \approx 37°C$ ← that's us, cats are like 38 or 39°

$\boxed{5}$ a) $-2+2i = 2(-1+i)$

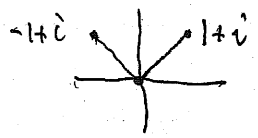
$= 2\sqrt{2}\, e^{3\pi/4\, i}$

Principal cube root is

$= 2^{3/2}\, e^{3\pi/4\, i}$

$\sqrt[3]{-2+2i} = 2^{1/2}\, e^{\pi/4\, i}$

$= 1+i$



b) $\tilde{y}'' = e^{(1+2i)t}$ Want $\mathrm{Im}(\tilde{y})$

Integrate twice, get

$$\tilde{y} = \frac{e^{(1+2i)t}}{(1+2i)^2} = \frac{e^{t}\cdot e^{2it}}{-3+4i}$$

$$\tilde{y} = \frac{-3-4i}{25}\cdot e^{t}\cdot(\cos 2t + i\sin 2t)$$

$$\mathrm{Im}(\tilde{y}) = e^{t}\left(\frac{-3}{25}\sin 2t - \frac{4}{25}\cos 2t\right)$$

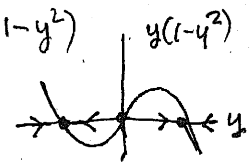
$\boxed{6}$ a) $y' = y - y^3 = y(1-y^2)$

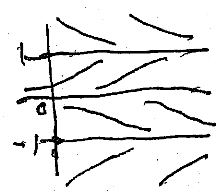
Zeros are at $0, \pm 1$

from picture:

0 unstable

1, -1 stable





solution curves

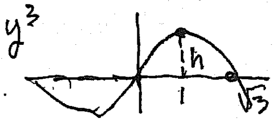
b) want 3 to be a stable zero of $ay - y^3$

$\therefore a\cdot 3 - 3^3 = 0$ so $\boxed{a=9}$

(picture is essentially same as above, $\therefore$ stable crit. pt.)

c) $y' = 3y - y^3$

$y(3-y^2)$



zeros: $0, \pm\sqrt{3}$ To find h:

$$\frac{d}{dy}(3y - y^3) = 3 - 3y^2 = 0 \text{ at } y = \pm 1$$

$$3y - y^3\big|_1 = 2 = h$$

$y' = 3y - y^3 - 2 = f(y)$