Linear Phase Portraits*

18.03 Class 35

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Vocabulary: $(\operatorname{tr} A, \det A)$ plane, critical parabola, spiral, node, saddle; center, star, defective node; degeneracy; stable, asymptotic, neutral; unstable.

The moral of today's lecture: Eigenvalues Rule (usually).

Recall that the characteristic polynomial of a square matrix A is $p_A(\lambda) = \det(A - \lambda I)$. When A is a 2×2 matrix, we can write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so the characteristic polynomial of A can be rewritten as

$$p_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + (\det A)$$

where $\operatorname{tr} A = a + d$, $\det A = ad - bc$.

The roots of the characteristic polynomial p_A are the eigenvalues of A, so if we denote the eigenvalues by λ_1 and λ_2, we can write

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + (\lambda_1\lambda_2)$$

Comparing coefficients, we see that

$$\operatorname{tr} A = \lambda_1 + \lambda_2 \qquad \det A = \lambda_1\lambda_2$$

so the two numbers $\operatorname{tr} A$ and $\det A$—calculated from the entries a, b, c, and d of the matrix A—are determined by the eigenvalues. Conversely, $\operatorname{tr} A$ and $\det A$ determine the eigenvalues as the roots of the quadratic quation, namely

$$\lambda_1, \lambda_2 = \frac{\operatorname{tr} A}{2} \pm \sqrt{\frac{(\operatorname{tr} A)^2}{4} - \det A}.$$

We can see that the eigenvalues λ_1 and λ_2 are

not real if $\det A > (\operatorname{tr} A)^2/4$,

*Taken from the 18.03 website and edited slightly. Please let `lee@math` know if there are mistakes.

equal if $\det A = (\operatorname{tr} A)^2/4$, and

real and distinct if $\det A < (\operatorname{tr} A)^2/4$.

The dividing line is the *critical parabola*, where $\det A = (\operatorname{tr} A)^2/4$.

Notice that if the eigenvalues are complex, their real part is $(\operatorname{tr} A)/2$. If the eigenvalues are real, they have the same sign exactly when their product is positive, and that sign is positive if their sum is also positive. Thus:

```
                         det

                          ^
    .                     |<-----purely imaginary
     .            complex roots                 .
      .                   |                    .
       .      Re < 0      |     Re > 0       .<----- repeated
         .           |                  .
           .              |                .
   real < 0  .            |             .     real > 0
               ..         |          ..
                   ..     |     ..
  ------------         .......         ------------>  tr
                                             ^
                                             |
                 real, opposite sign         |______  at least one zero e.v.
```

The corresponding phase portraits exhibit the following behaviors:

```
  stars or               det
  defective nodes
     |                    ^
    .V                    |<-----centers       .
     .                    |                   .
      .    stable         |   unstable       .
       .   spirals        |   spirals       .
         .           |                  .
           .              |                .
   stable    .            |             .   unstable
   nodes       ..         |          ..     nodes
                   ..     |     ..
  ------------         .......         ------------>  tr
                                             ^
                                             |
                       saddles               |______  degenerate cases
```

The only important part of this classification I haven't discussed is the *nodes.*

Example: $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$.

This matrix is *upper triangular*, meaning that it is zero below the main diagonal. In this case the eigenvalues are dead easy to read off; they are precisely the diagonal entries. This because the eigenvalues are characterized by the fact that their sum is the trace and their product is the determinant, and, because of the 0, this is also true of the diagonal entries.

So the eigenvalues here are $\lambda_1 = 2$, $\lambda_2 = 1$, so

- for $\lambda_1 = 2$, $(A - \lambda_1 I)\vec{\alpha}_1 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \vec{\alpha}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\vec{\alpha}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,
- for $\lambda_2 = 1$, $(A - \lambda_2 I)\vec{\alpha}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \vec{\alpha}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\vec{\alpha}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

so the normal modes are $e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (remember that we can take any nonzero multiple of $\vec{\alpha}_1$ or $\vec{\alpha}_2$).

The ray solutions are along the x-axis and along the line with slope 1. Both increase exponentially in size, but the $\lambda_1 = 2$ eigensolution blows up like the square of the $\lambda_2 = 1$ solution.

The general solution is a linear combination of these two. Take the sum, for example; then $u(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. When $t \ll 0$, the e^{2t} term is much smaller than the e^{t} term, and so the linear combination is very near the $\lambda_2 = 1$ eigenline, namely the line of slope 1. The result is a spider like figure, with body along that line and legs opening out from it. This picture is shown on p. 90 of the Supplementary Notes. This is an *unstable node.*

There are also the special cases that happen along the curves separating these regions:

- $\operatorname{tr} A = 0$ and $\det A > 0$: eigenvalues are nonzero and purely imaginary. The phase portraits are *centers.* All trajectories (except the constant solution at the origin) are ellipses.
- $\det A = 0$: at least one of the eigenvalues is zero. If $\vec{\alpha}$ is an eigenvector corresponding to this eigenvalue, then the constant vector valued function $u(t) = c\vec{\alpha}$ is a solution for any constant c; there is (at least) a line of constant solutions. Several patterns are possible, and they are illustrated in the Supplementary Notes.

- $\det A = (\operatorname{tr} A)^2/4$, along the critical parabola: repeated real eigenvalues. The phase portraits are either stars, in the complete case $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$ or defective nodes, otherwise.

The phase portrait in each one of these borderline cases shows some features which are not determined purely by the eigenvalues. In addition to these: when $\det A > (trA)^2/4$, the phase portrait is made up of spirals, but you can't tell from the eigenvalues alone which way the spiral is rotating. To discover that in case $A = \begin{bmatrix} -2 & 5 \\ -2 & 4 \end{bmatrix}$ for example, let's just evaluate the vector field at the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; it is given by the first column of A, $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$, and so the vector field points south (and west) at this point, and the rotation is counterclockwise.

Stability: All linear systems fall into one of the following categories:

Asymptotically stable: all solutions tend to zero as $t \to \infty$. These systems occupy the upper left quadrant, $\operatorname{tr} A < 0$ and $\det A > 0$, that is, the eigenvalues have negative real part.

Neutrally stable: all solutions are periodic. These systems occur only along the ray $\operatorname{tr} A = 0$, $\det A > 0$, so the eigenvalues are nonzero and purely imaginary. In these linear cases the nonzero trajectories are in fact ellipses.

Unstable: most solutions tend to infinity as $t \to \infty$. Saddles and unstable nodes and spirals are examples.

At the end of class I showed an animation of the way the phase portraits change as you move around a loop in the $(\operatorname{tr} A, \det A)$ plane. For each pair $(\operatorname{tr} A, \det A)$, the companion matrix

$$\begin{bmatrix} 0 & 1 \\ -\det A & -\operatorname{tr} A \end{bmatrix}$$

provides an example of a matrix with this trace and determinant. The corresponding phase portraits are illustrated. This animation can be found at `http://www.awlonline.com/ide/` under *Linear Algebra* → *Linear Classification* → *Parameter Path Animation Tool*. You might look at other applets in this collection.

There is also the Mathlet LinearPhaseCursor that gives a good representation of the variety of phase portraits with given $(\operatorname{tr} A, \det A)$ pair. This actually exists at `http://www-math.mit.edu/~ashot/LinearParameters.htm` as a java program.